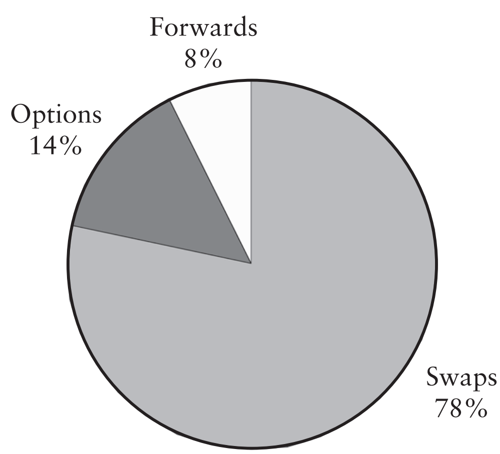
CHAPTER 18

Interest Rate Products: Swaps

The first interest rate swap market originated in the early 1980s. An *interest rate swap* is an agreement between two parties to exchange or "swap" a series of periodic interest payments. The most common interest rate swap, a *plain-vanilla interest rate swap*, is an agreement to exchange payments on fixed rate debt for floating rate debt. An early example occurred in 1982 when Sallie Mae swapped the interest payments on intermediate-term fixed rate debt for floating rate payments indexed to the three-month T-bill yield. In the same year, a USD 300 million seven-year Deutsche Bank bond issue was swapped into USD LIBOR. While we discussed swaps on other types of assets in earlier chapters, interest rate swaps are far and away the largest asset category. As of yearend 2003, interest rate derivatives accounted for 72% of the notional amount of all OTC derivatives outstanding. Of this amount, more than 78% of interest rate derivatives were swaps, with the remaining 22% being split between options (14%) and forwards (8%) as is shown in Figure 18.1.

FIGURE 18.1 Percentage of total notional amount of single-currency interest-rate derivatives outstanding worldwide on December 2003 by contract type. Total notional amount of interest-rate derivatives is USD 141.99 trillion.



Source: The table was constructed from information contained in Bank for International Settlements (www.bis.org), *BIS Quarterly Review*, June 2004.

In general, this chapter deals with OTC interest rate products that have multiple cash flows through time. While plain-vanilla swaps is certainly the largest category within this group, there are also a variety of other instruments including caps, collars, floors, and swaptions. We will address each in turn. Before doing so, however, it is important to develop a thorough understanding of the zero-coupon yield curve and how it is estimated. This is the focus of the first section of this chapter. The second section describes the nature of interest rates swaps and how they are valued. The third and fourth sections focus on caps, collars, and floors, and swaptions, respectively.

ESTIMATING THE ZERO-COUPON YIELD CURVE

In Chapter 2, we defined the term structure of interest rates (or the zero-coupon yield curve) as the relation between yield and term to maturity for zero-coupon bonds with a common degree of default risk. At the time, we used U.S. Treasury bills and strip bonds to illustrate the shape of the term structure. In the illustrations of the chapters that followed, we assumed that we knew the structure of the zero-coupon yield curve and usually expressed it as some form of mathematical function such as, for example, $r_i = 0.03 + 0.01\ln(1 + T_i)$. The assumption was motivated by the need to have a risk-free, zero-coupon interest rate for all future dates on which there was a cash flow.[1] In this section, we face the problem of determining the zero-coupon yield curve head on.

Estimating the zero-coupon yield curve has two steps. First, we must collect prices of instruments with varying times to maturity but the same degree of default risk. These are usually either U.S. Treasury rates or Eurodollar rates. Within each of these categories, we must choose among available instruments. For U.S. Treasuries, for example, the zero-coupon rates can be estimated using any combination of T-bills, strips, coupon-bearing notes and bonds, and constant maturity Treasury (CMT) rates. For Eurodollars, time-deposit rates, futures prices, and swap rates can all be used. The choice depends on the application at hand and the liquidity of the markets whose rates/prices are being used. From the prices of these instruments, we determine zero-coupon yields for terms to maturity, as is illustrated in Figure 18.2.

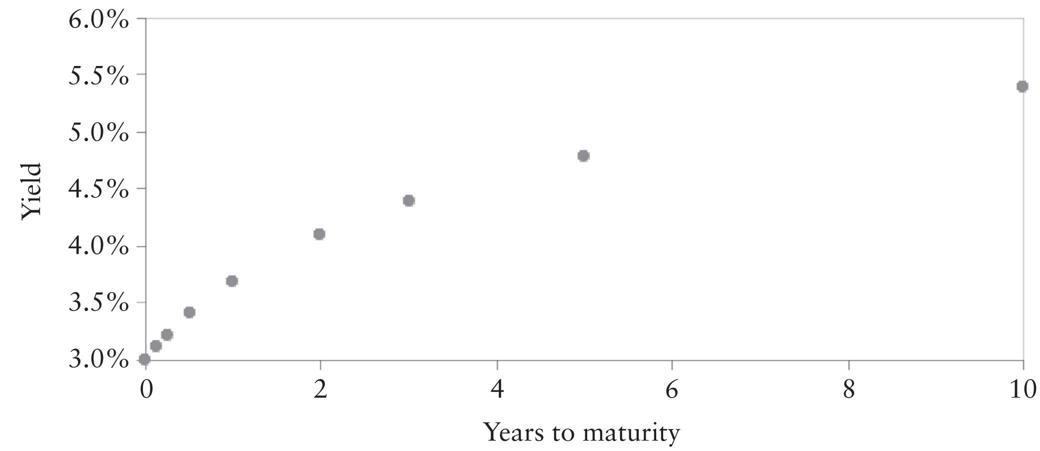
The second step involves "smoothing the yield curve." More specifically, we must decide how to estimate zero-coupon rates for cash flows that occur at times other than those represented in Figure 18.2. Consider a cash flow that occurs four years from now. We have only a zero-rate for year three and one for year five. Based on these rates, or any other rates in the figure, what is the best guess of the four-year rate? We discuss two possible methods.

Identify Zero-Coupon Rates

As was noted earlier, zero-coupon yield curves are most commonly constructed using either U.S. Treasury rates or Eurodollar rates. Below we focus first on estimating the yield curve for Treasuries and then for Eurodollars.

[1] Put differently, we need to know today's prices of one dollar received at all future cash flow dates. These, of course, are the discount factors implied by the zero-coupon yield curve.

FIGURE 18.2 Zero-coupon rates determined from available interest rate securities and derivative contracts.



Treasury Instruments U.S. Treasury instruments come in a variety of forms. First, at the short-end, there are T-bills. T-bills are discount instruments and therefore have no intermediate interest payments. Each week the U.S. Treasury issues 28-day, 91-day, and 182- T-bills. As noted in Chapter 2, T-bill rates are quoted as a discount from par and use a 360-day banker's year. To compute the continuously-compounded yield to maturity of these discount instruments, we use the formula,

$$r_i = \frac{\ln(100/B_i)}{T_i} \tag{18.1}$$

where B_i is the price of the T-bill, which is determined by taking the bill's quoted discount, D_i, and adjusting it by the number of days to maturity, n_i, in the following way,

$$B_i = 100 - D_i(n_i/360) \tag{18.2}$$

and T_i is the actual number of years to maturity (i.e., $T_i = n_i/365$). Thus, based on quoted T-bill discount rates, we can identify the dots in Figure 18.2 up until 182 days to maturity.

To go beyond six months, however, we have a variety of alternatives. Strip bonds would be ideal since they have no intermediate coupon payments and their prices can be transformed to zero-coupon interest rates quite easily using (18.1), where B_i represents the strip bond price quote. Unfortunately, the market for strip bonds is not particularly active, so the quote prices are sometimes unreliable. The same is true for most coupon-bearing notes and bonds. Most Treasuries trade actively for a short period of time just after they are issued (called *on-the-run* issues), and then infrequently thereafter (*off-the-run* issues).

In practice, the zero-coupon yield curve for Treasuries is usually constructed from "Constant Maturity Treasury" rates, or CMTs. CMT yields are computed each day by the U.S. Treasury and are intended to represent the yields to maturity of par bonds[2] with 1, 3 and 6 months and 1, 2, 3, 5, 7, 10, and 20 years to maturity. To estimate these rates, the Treasury gathers the closing market bid yields on on-the-run Treasury securities. These market yields are calculated from composites of over-the-counter market quotations obtained by the Federal Reserve Bank of New York each day. Based on these market yields, the Treasury then smoothes the relation between yield and term to maturity,[3] thereby allowing it to estimate rates at the standard maturities listed above. Thus a yield for a 10-year maturity can be computed even if no outstanding security has exactly 10 years remaining to maturity. To generate the zero-coupon yields for all maturities, we "reverse engineer" the CMTs using a technique called "bootstrapping."

In estimating the zero-rates from CMT rates, we must first separate CMTs into two groups—those with maturities of less than a year and those with maturities one year or greater. The reason is that short-term CMTs have no coupon interest payments while the long-term ones do. The following table shows the rates observed as of the close of trading on March 17, 2005. They were obtained from the U.S. Treasury's website at http://www.ustreas.gov/offices/domestic-finance/debt-management/interest-rate/yield.html. These rates will serve as the basis for illustrating the bootstrapping technique as we proceed with its description.

CMT Rates

Type	Term	Yield
Months	1	2.68
	3	2.79
	6	3.08
Years	1	3.29
	2	3.70
	3	3.89
	5	4.14
	7	4.30
	10	4.47
	20	4.87

To begin, we find the zero-coupon rates corresponding to the 1, 3, and 6 month CMT rates. As was noted above, these are not coupon bonds. There is one payment at the end of the bond's life that includes coupon interest as well as the repayment of principal. The continuously compounded zero-coupon yield to maturity for each of these three CMTs can be computed using

$$r_i = \frac{\ln(1 + (y_i/100)(m_i/12))}{m_i/12} \tag{18.3}$$

[2] A par bond is one whose price equals its face value. For such a bond, the coupon interest rate equals its yield to maturity compounded on a semiannual basis.
[3] The Treasury uses a cubic spline model to smooth the yield curve.

where y_i is the annualized nominal yield to maturity of CMT$_i$, and m_i is its number of months to maturity. For the one-month maturity, for example, the zero-coupon rate is

$$r_i = \frac{\ln(1 + (2.68/100)(1/12))}{1/12} = 2.677\%$$

The rates for three months and six months, together with the one-month rate, are summarized as follows:

CMT Rates			
Type	**Term**	**Yield**	**Zero-Rate**
Months	1	2.68	2.677%
	2	2.79	2.780%
	3	3.08	3.057%

Now we turn to the coupon-bearing CMT rates. Matters get slightly more complicated. For maturities of one year and greater, the CMT rates are yields on par bonds with semiannual coupon payments. A par bond is one whose price equals its face value. For such a bond, the coupon interest rate equals its yield to maturity compounded on a semiannual basis. This means that each CMT bond may be written as

$$100 = \sum_{i=1}^{n-1} e^{-r_i T_i}\left(\frac{COUP}{2}\right) + e^{r_n T_n}\left(100 + \frac{COUP}{2}\right) \tag{18.4}$$

where r_i is the zero-coupon rate of a bond maturing at time T_i, $COUP$ is the annualized coupon rate (i.e., the CMT rate) of the bond under consideration, and n is its number of semiannual coupons. What the bootstrapping technique does is start with the zero-coupon rate at the shortest maturity, and then solve for each new maturity recursively one at a time. Consider the one-year CMT rate. A one-year semiannual coupon CMT reported in the panel above has a yield of 3.29%. Substituting into (18.4), we get

$$100 = \sum_{i=1}^{2-1} e^{-0.03057(0.5)}\left(\frac{3.29}{2}\right) + e^{r_2(1)}\left(100 + \frac{3.29}{2}\right)$$

In this expression, the first term on the right-hand side is the present value of the first semiannual coupon which we can compute because we have already determined that the six-month continuously compounded zero-rate is 3.057%. Since we have one equation and one unknown, we can solve for the one-year zero-coupon rate by rearranging the expression to isolate r_2. Its value is 3.265%.

The next available CMT rate has two years to maturity. Substituting into (18.4) we get

$$100 = e^{-0.03057(0.5)}\left(\frac{3.70}{2}\right) + e^{-0.03265(1)}\left(\frac{3.70}{2}\right) + e^{-r_3(1.5)}\left(\frac{3.70}{2}\right) + e^{r_4(2)}\left(100 + \frac{3.70}{2}\right)$$

Now we are in a pickle. We have one equation and need to solve for the 1.5-year and two-year zero-coupon rates. To manage this particular conundrum, we assume that the 1.5 year rate equals the average of the one-year rate and the two-year rate, that is,

$$r_3 = \frac{r_2 + r_4}{2} = \frac{0.03265 + r_4}{2}$$

By imposing this restriction, we can compute r_4 and, hence, r_3. The two-year zero-rate is 3.676%, and the 1.5-year zero-rate is 3.470%.

From a practical perspective, it is best to go ahead and compute the CMT rates at half year interval from the outset. With the one-year CMT rate at 3.29% and the two-year CMT rate at 3.70%, the 1.5-year CMT rate, computed using linear interpolation, is 3.495%. With the three-year CMT rate at 3.89% and the five-year CMT rate at 4.14%, the 3.5-year CMT rate, computed using linear interpolation, is 3.9525%, and so on. Now, the zero-coupon rates at half-year intervals from one year to 20 years can be determined recursively (i.e., "bootstrapped") one at a time using a re-arranged version of equation (18.4), that is,

$$r_n = \frac{\ln\left(\dfrac{100 - \sum_{i=1}^{n-1} e^{-r_i T_i} COUP/2}{100 + COUP/2}\right)}{T_n} \tag{18.5}$$

The last rate we are able to compute has the same term to maturity as the longest CMT rate.

The bootstrap procedure for deducing zero-coupon rates from CMT rates is programmed as a function in the OPTVAL Function library. Its syntax is

OV_IR_TS_ZERO_FROM_CMT(*months*, *cmtm*, *years*, *cmty*, *rt*)

where *months* is the vector of months to maturity of the CMT rates less than a year, *cmtm* is the vector of rates of the CMT rates less than a year, *years* is the vector of years to maturity of the CMT rates one year or greater, *cmty* is the vector of rates of the CMT rates one year or greater, and *rt* is an indicator variable set to *r* or *R* if the function is to return an array of zero-coupon rates or *t* or *T* if the function is to return an array of the years to maturity of the zeros.[4] To use the function, we highlight cells F3:F14, call the function OV_IR_TS_ZERO_FROM_CMT and insert the necessary inputs, and then press Shift, Ctrl, and Enter simultaneously. The high-

[4] The function returns rates/terms corresponding to the maturities of the monthly input rates and then at half year intervals thereafter.

lighted region will then fill with the zero-coupon rates. Note that the 1.5-year and two-year rates correspond to our computations above.

F3 {=OV_IR_TS_ZERO_FROM_CMT(B3:B5,C3:C5,B6:B9,C6:C9,"R")}

	A	B	C	D	E	F	G	H	I
1	CMT rates					Zero-			
2	Type	Term	Yield		Years	rate			
3	Months	1	2.68		0.0833	0.02677			
4		3	2.79		0.2500	0.02780			
5		6	3.08		0.5000	0.03057			
6	Years	1	3.29		1.00	0.03265			
7		2	3.70		1.50	0.03470			
8		3	3.89		2.00	0.03676			
9		5	4.14		2.50	0.03771			
10					3.00	0.03867			
11					3.50	0.03930			
12					4.00	0.03994			
13					4.50	0.04059			
14					5.00	0.04124			

Eurodollars For Eurodollars, zero-coupon rates are usually estimated using either (1) Eurodollar time-deposit rates for maturities less than one year and Eurodollar swap rates for one year and beyond; or (2) Eurodollar time-deposit rates for maturities to three months and Eurodollar futures prices beyond three months. If a combination of time-deposit and swap rates is used (approach (1)), the bootstrapping technique described for the CMT rates can be applied once again. Time-deposit rates are nominal interest rates on short-term deposits where interest is paid only at maturity, and swaps rates are essentially the coupon rates of semiannual coupon par bonds. On March 17, 2005, Eurodollar time deposit and swap rates were as follows:

Eurodollar Time Deposits		Eurodollar Swap Rates	
Months	Rate	Years	Rate
1	2.8281	1	3.6900
3	3.0156	2	4.0800
6	3.2656	3	4.2950
		4	4.4400
		5	4.5550
		6	4.6400
		7	4.7150
		8	4.7850
		9	4.8500
		10	4.9050
		12	5.0000
		15	5.1050
		20	5.2000
		25	5.2350
		30	5.2500

The bootstrap procedure for deducing zero-coupon rates from Eurodollar time deposit/swap rates is programmed as a function in the OPTVAL Function library. Its syntax is

OV_IR_TS_ZERO_FROM_SWAP(*months*, *spot*, *years*, *swap*, *rt*)

where *months* is the vector of months to maturity of the time-deposit rates with maturities less than a year, *spot* is the vector of time-deposit rates, *years* is the vector of years to maturity of the swap rates of one year or greater, *swap* is the vector of swap rates, and and *rt* is an indicator variable set to *r* or *R* if the function is to return an array of zero-coupon rates or *t* or *T* if the function is to return an array of the years to maturity of the zeros.[5] To use the function, we highlight cells H3:H14, call the function OV_IR_TS_ZERO_FROM_SWAP and insert the necessary inputs, and then press Shif, Ctrl, and Enter simultaneously. The highlighted region will then fill with the zero-coupon rates.

H3 fx {=OV_IR_TS_ZERO_FROM_SWAP(A3:A5,B3:B5,D3:D7,E3:E7,"R")}

	A	B	C	D	E	F	G	H	I
1	Eurodollar time deposits			Eurodollar swap rates				Zero-	
2	Months	Rate		Years	Rate		Years	rate	
3	1	2.8281		1	3.6900		0.0833	2.825%	
4	3	3.0156		2	4.0800		0.25	3.004%	
5	6	3.2656		3	4.2950		0.50	3.239%	
6				4	4.4400		1.00	3.660%	
7				5	4.5550		1.50	3.854%	
8							2.00	4.050%	
9							2.50	4.158%	
10							3.00	4.267%	
11							3.50	4.341%	
12							4.00	4.415%	
13							4.50	4.474%	
14							5.00	4.534%	

Note that we have computed continuously-compounded, zero-coupon rates for Treasuries and Eurodollars with comparable maturities and that the Eurodollar rates are uniformly higher. This reason is simple—credit risk. While both are rates of return on U.S. dollar deposits, Treasury rates are backed by the resources of the U.S. government. Eurodollar rates, on the other hand, are banked by the resources of the British bank where the deposit is held. Note also that the credit risk premium grows larger with term to maturity. This reflects the fact that the probability of default increases with time. While there may be little chance that the bank will default during the next year, there may be a significantly larger risk that it will default over the next 30 years.

[5] This function also returns rates/terms corresponding to the maturities of the monthly input rates and then at half year intervals thereafter.

	Zero-Coupon Rates		
Years	Treasuries	Eurodollars	Risk Premium
0.0833	2.677%	2.825%	0.148%
0.25	2.780%	3.004%	0.224%
0.50	3.057%	3.239%	0.183%
1.00	3.265%	3.660%	0.395%
1.50	3.470%	3.854%	0.385%
2.00	3.676%	4.050%	0.374%
2.50	3.771%	4.158%	0.387%
3.00	3.867%	4.267%	0.400%
3.50	3.930%	4.341%	0.410%
4.00	3.994%	4.415%	0.421%
4.50	4.059%	4.474%	0.415%
5.00	4.124%	4.534%	0.410%

In the Eurodollar market, zero-coupon yields are also often computed using a combination of Eurodollar time-deposit rates and Eurodollar futures prices. The procedure is not unlike the bootstrapping procedure using with CMT and swap rates in the sense that we start with the shortest term to maturity and then add longer maturities, one at a time.[6] First, we identify the rate of interest on a Eurodollar time deposit that matures when the nearby quarterly Eurodollar futures contract settles. Standing on March 17, 2005, the nearby quarterly June futures expires June 15, 2005—in 90 days. The three-month Eurodollar time deposit rate was given earlier in this section and is 3.1056%. The continuously compounded, zero-coupon rate for this maturity is therefore

$$r_{90} = \frac{\ln\left(1 + 0.031056\left(\frac{90}{360}\right)\right)}{90/365} = 3.0460\%$$

Next, we use the settlement price of the June 2005 Eurodollar futures contract, 96.1510 to compute the forward rate on a Eurodollar time deposit that begins on June 15, 2005 and ends when the September 2005 settles on September 21, 2005. The forward rate expressed as a nominal rate is 100 − 96.5150 = 3.4850%. Expressed as a continuously compounded rate, the implied forward rate on a 98-day time deposit beginning in 90 days is

$$r_{98,90} = \frac{\ln\left(1 + 0.034850\left(\frac{98}{360}\right)\right)}{98/365} = 3.5167\%$$

[6] The procedure described here is intended to be illustrative only. We ignore considerations such as two-day settlement, three-month time intervals with varying numbers of days, and convexity.

The 188-day continuously compounded, zero-coupon rate is therefore determined by

$$r_{188}\left(\frac{188}{365}\right) = 0.03460\left(\frac{90}{365}\right) + 0.03517\left(\frac{98}{365}\right)$$

and is 3.2914%. The panel below summarizes the computations out to five years to maturity. The syntax of the OPTVAL function is

OV_IR_TS_ZERO_FROM_EDFUT(*ndt, srate, nexp, fp, rt*)

where *ndt* is today's date, *srate* is the rate of interest on the time deposit maturing when the nearby futures contract settles, *nexp* is the vector of settlement dates for the Eurodollar futures, *fp* is the corresponding vector of futures prices, and *rt* is an indicator variable instructing the function to return the term of maturity, *T*, or the zero-coupon rate, *R*.

F3 fx {=OV_IR_TS_ZERO_FROM_EDFUT(A3,C3,B8:B27,D8:D27,"T")}

	A	B	C	D	E	F	G	H
1	Eurodollar time deposit						Zero-	
2		Days	Rate			Years	rate	
3	3/17/2005	90	3.0156			0.2466	3.0460%	
4						0.5151	3.2914%	
5	Eurodollar futures prices					0.7644	3.4931%	
6	Contract	Settlement	Days until	Settlement		1.0137	3.6585%	
7	month	date	next	price		1.2630	3.7873%	
8	Jun-05	6/15/2005	98	96.5150		1.5123	3.8927%	
9	Sep-05	9/21/2005	91	96.1250		1.7616	3.9810%	
10	Dec-05	12/21/2005	91	95.8700		2.0110	4.0574%	
11	Mar-06	3/22/2006	91	95.7250		2.2603	4.1218%	
12	Jun-06	6/21/2006	91	95.6100		2.5096	4.1785%	
13	Sep-06	9/20/2006	91	95.5200		2.7589	4.2299%	
14	Dec-06	12/20/2006	91	95.4400		3.0082	4.2777%	
15	Mar-07	3/21/2007	91	95.3950		3.2575	4.3210%	
16	Jun-07	6/20/2007	91	95.3450		3.5068	4.3612%	
17	Sep-07	9/19/2007	91	95.2900		3.7562	4.3992%	
18	Dec-07	12/19/2007	91	95.2300		4.0055	4.4361%	
19	Mar-08	3/19/2008	91	95.1950		4.2548	4.4710%	
20	Jun-08	6/18/2008	91	95.1500		4.5041	4.5044%	
21	Sep-08	9/17/2008	91	95.1050		4.7534	4.5365%	
22	Dec-08	12/17/2008	91	95.0450		5.0027	4.5682%	
23	Mar-09	3/18/2009	91	95.0050				
24	Jun-09	6/17/2009	91	94.9650				
25	Sep-09	9/16/2009	91	94.9200				
26	Dec-09	12/16/2009	91	94.8650				
27	Mar-10	3/17/2010		94.8350				

Smoothing the Yield Curve

Thus far we have performed the first step in identifying the zero-coupon yield curve, that is, we have identified a series of zero-coupon spot rates at specific maturities (i.e., we have identified the location of the dots in Figure 18.2). The next step in building the zero-coupon yield curve involves deciding how to a zero-coupon rate at a maturity that falls between the dots in Figure 18.2. Suppose, for example, a cash flow that occurs four years from now. We have only a zero-rate for year three and one for year five. What is the best guess of the four-year rate? We discuss two possible methods.

Perhaps, the most popular method for handling this problem is called *linear interpolation*.[7] In essence, it involves drawing a straight line between the two rates on the term structure that straddle the desired maturity, and then reading the rate from the line. Algebraically, this amounts to the time-weighted average,

$$r_k = r_i\left(\frac{T_j - T_k}{T_j - T_i}\right) + r_j\left(\frac{T_k - T_i}{T_j - T_i}\right) \tag{18.6}$$

where i and j are the rates on either side of the desired maturity k, T_m is the time to maturity of the mth rate (measured in days or years), and $T_i \le T_k \le T_j$. Suppose we would like to determine the six-month zero-coupon rate based on the zero-coupon rates we computed from futures prices. Applying the formula (18.6), we get

$$r_k = 0.030460\left(\frac{0.5000 - 0.2466}{0.5151 - 0.2466}\right) + 0.032914\left(\frac{0.5151 - 0.5000}{0.5151 - 0.2466}\right) = 3.278\%$$

In the event T_k is less (greater than) $T_i(T_j)$, r_k is set equal to $r_i(r_j)$. Linear interpolation can be performed using the function,

OV_IR_TS_INTERPOLATE(*sterm,term,rate*)

where *sterm* is the term to maturity of the desired rate, *term* is a vector of the terms to maturity of the available rates, and *rate* is the vector of available rates.

Another smoothing technique involves fitting a regression line through the available zero-coupon points. Suppose, for example, we fit the regression

$$r_i = \alpha_0 + \alpha_1 \ln(1 + T_i) + \varepsilon_i$$

through the zero-coupon rates deduced from Eurodollar futures prices. The fitted regression line is

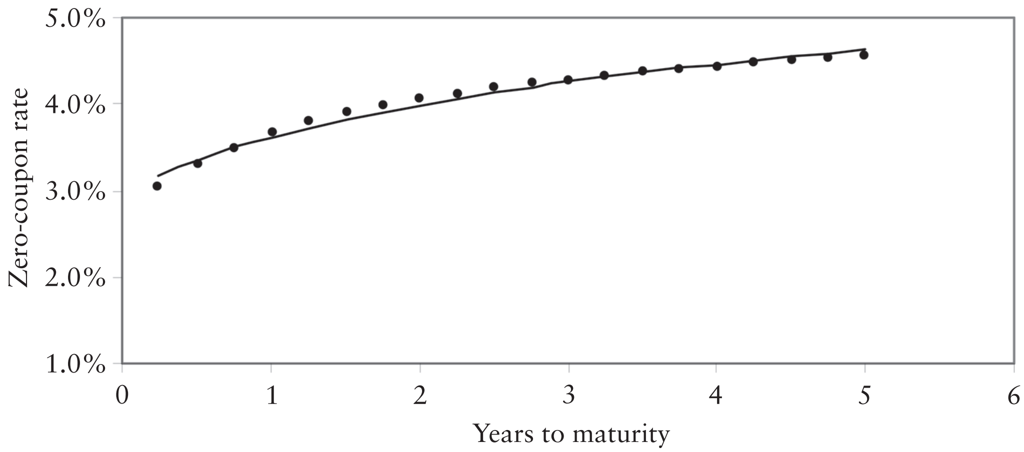
$$\hat{r}_i = 0.02976 + 0.00922 \ln(1 + T_i)$$

As the figure below show, the regression does reasonably well at smoothing the points, with a tendency to overestimate very short-term rates and underestimate

[7] Other smoothing methods include multivariate regression and cubic splines. For a detailed description of different curve-fitting methods, see Tuckman (2002, Ch. 4).

intermediate term rates. The regression estimate for the six-month, zero-coupon rate is 3.438%.

Using a more elaborate regression model structure would improve matters. Two simple alternatives are to express the zero-coupon rate as a quadratic or cubic function of time to maturity. Regardless, however, the regression approach is somewhat troublesome in the sense that it will generally produces predicted zero-coupon rates that are different from the rates that are used as inputs in the regression. Put differently, the line does not go through the points in the figure below. Under linear interpolation, this would never happen.



INTEREST RATE SWAPS

The specifications of OTC interest rate swap contracts are much less transparent than for interest rate futures and options traded on exchanges. The reason is simple. The contracts are privately negotiated between counterparties, with neither having any obligation to report the terms publicly. Thanks to trade organizations such as the International Swaps and Derivatives Association (or *ISDA*), certain standard practices have emerged. Documents such as *2000 ISDA Definitions* and *Annex to the 2000 ISDA Definitions*[8] lay out the industry's "language" for communicating the terms of derivatives transactions. Other documents such as the *ISDA Master Agreement (Local Currency – Single Jurisdiction)* and the *ISDA Master Agreement (Multicurrency – Cross Border)* provide the text for actual contracts.

In this chapter, we focus primarily on plain-vanilla interest rate swaps. In these swaps, one leg requires the payment of interest based on a fixed rate, and the other leg requires payment of interest based on a floating rate. These swaps have become so active that their rates are quoted widely, and the spread between bid and ask rates is as little as four basis points. Table 18.1 contains midmarket fixed-for-floating swap rates as of the close of trading on Friday January 28, 2005. These rates are for "generic" interest rate swaps. Specifically, for these swaps, (1) no money changes hand at inception; (2) no exchange of principal

[8] See International Swap and Derivatives Association's (ISDA's) website at www.isda.org.

TABLE 18.1 Fixed-for-floating swap rates reported by Bloomberg on Friday, January 28, 2005. By convention, the rates represent the fixed rate on a swap with semiannual interest payments and a floating rate based on six-month LIBOR. A swap to receive fixed and pay floating will be based on the bid rate, and a swap to pay fixed and receive floating will be based on the ask rate.

Term in Years	Bid	Ask
2	3.557	3.589
3	3.753	3.784
4	3.906	3.938
5	4.052	4.060
6	4.152	4.186
7	4.250	4.285
8	4.341	4.376
9	4.422	4.458
10	4.493	4.529
15	4.761	4.796
20	4.888	4.926
30	4.961	4.999

occurs; (3) interest payments are made semiannually and are netted (i.e., the party owing the largest payment pays the difference between the amount he owes and the amount he is supposed to receive); and (4) the floating rate is based on the six-month LIBOR rate.[9] With the contractual terms in mind, we can now interpret Table 18.1. The table contains the fixed rate on a fixed-for-floating swap. Thus, for a two-year fixed-for-floating swap, the fixed rate payer pays 3.589/2 or 1.7945% and receives six-month LIBOR rate each six months.

Fixed-for-floating interest rate swaps are just that—one party agrees to pay a fixed rate of interest and receive a floating rate, and the other party receives a fixed rate of interest and pays a floating rate. Interest rate swaps are usually consummated by a confirmation sheet faxed between the counterparties in the OTC market. Table 18.2 shows selected terms from a confirmation sheet of a plain-vanilla interest rate swap. The sheet is divided into three panels of information. The first panel provides the calculation amount, trade date, and termination date. The *calculation amount* is the notional amount upon which interest payments are computed. The *trade date* is the day on which the parties enter

[9] In some swaps, the interest rate on the floating rate leg gets reset more frequently than the payments (e.g., the floating-rate gets reset each month based on one-month LIBOR while interest payments are made semiannually). In these instances, the one-month reset rates observed during the payment interval are averaged to determine the floating rate payment. In general, the swap agreement will specify the method of averaging as "unweighted" or "weighted." Unweighted means a simple arithmetic average of all rates during the payment interval, and weighted means a time-weighted arithmetic average (i.e., each set rate is weighted by the proportion of the total number of days that the rate prevailed during the payment period). If the term sheet does not specify the method of averaging, unweighted averaging is assumed. See International Swaps and Derivatives Association (2000b, p.9).

TABLE 18.2 Selected terms from the confirmation of an OTC interest rate swap

The terms of the particular swap transaction to which this confirmation relates are as follows:	
Calculation amount	USD 30,000,000.00
Trade date	May 28, 2004
Effective date	June 1, 2004
Termination date	June 1, 2009
The fixed rate payer pays on each payment date an amount determined in accordance with the following:	
Fixed rate payer	Bank A
Payment dates	Commencing on December 1, 2004 and semiannually thereafter on the first calendar day of each calendar day of June and December up to and including the termination date.
Fixed rate	4.238%
Fixed rate, day-count fraction	30/360
The floating rate payer pays on each payment date an amount determined in accordance with the following:	
Floating rate payer	Company B
Payment dates	Commencing on December 1, 2004 and semiannually thereafter on the first calendar day of each calendar day of June and December up to and including the termination date.
Floating rate option	USD-LIBOR-LIBO
Designated maturity	6 months
Reset dates	The first day of the relevant calculation period
Rounding factor	One hundred-thousandth of 1%
Floating rate, day-count fraction	Actual/360

into the agreement, the *effective date* is the first day of the term of the agreement, and the *termination date* is the last day of the agreement.

The second and third panels of information specify obligations of the fixed-rate and floating rate payers, respectively. The fixed rate payer, in this case, is Bank A, which promises to make semiannual, fixed-interest payments at a rate of 4.238%. The "30/360" fixed rate, day-count fraction implies that each month (year) is assumed to have 30 (360) days. Thus Bank A is obliged to pay Company B an amount equal to

$$\$30,000,000 \times 0.04238 \times \frac{180}{360} = \$635,700$$

every six months for five years, with the first payment commencing on December 1, 2004.

At the same time, the floating rate payer, Company B, is obliged to make semiannual interest payments on the same dates. The *floating rate option* is specified to be "USD-LIBOR-LIBO" and the *designated maturity* is six months. The term, *USD-LIBOR-LIBO*, is defined in the *Annex to the 2000 ISDA Definitions*[10] and means the offered rate on U.S. dollar deposits for the period of the designated maturity as they appear on the Reuters Screen LIBO Page. Since the *reset date* is the first day of the calculation period, the first floating rate payment becomes known as of the effective date of the swap. If the rate is 1.5625% on June 1, 2004, the floating rate interest payment on December 1, 2004 will be computed as follows. First, you compute the actual number of days between June 1, 2004 and December 1, 2004. The actual number of days is 183. Next we compute the semiannual interest rate by taking the annual interest rate, 1.5625, multiplying it by the *floating rate, day-count fraction*, 183/360, and rounding it to 0.79427% (by virtue of the stated *rounding factor*). The floating rate payment that Company B is obliged to make on December 1, 2004 is $238,281. The fixed rate and floating rate payments are then *netted* so that only one party pays on a particular payment date. In our illustration, this means Bank A, the fixed rate payer, will pay Company B, the floating rate payer, $397,419 on December 1, 2004. Who pays and the amount of subsequent payments will depend on the level of the floating rates on the remaining reset dates.

In general, the terms of interest rate swaps are not available in financial publications such as the *Wall Street Journal*. Indeed, since OTC derivatives are privately negotiated and have wide-ranging terms, there are no means to systematically collect and report such information. One way to obtain indicative prices or rates of certain "generic" OTC derivatives deals is to subscribe to a service such as Bloomberg, Reuters, and Telerate that provides such quotes on a real-time basis. Essentially, what these services provide is access to a number of pages (computer screens), each page containing the current market quotes of generic types of trades. The fixed-for-floating swap rates shown in Table 18.1 are bid/ask quotes rates[11] from a real-time financial data service called *Bloomberg*. While interest rate swaps can have a wide variety of terms, the terms of these swaps are "standardized." The periodic payments of all these swaps are made semiannually, with the first payment occurring in six months. All of the rates are set in such a manner that the swaps have a zero upfront payment. The floating rate interest payment is indexed to the six-month LIBOR rate with an "actual/360" day-count fraction convention, and the fixed rate interest payment is based on the quotes appearing in the table and is calculated using a "30/360" day-count fraction convention. So, given these standard practices, the terms of the entire swap are summarized by the term and by the fixed rate. For real-time data services such as Bloomberg, bid and ask rates are displayed. These represent the highest bid rate and the lowest ask rate of all OTC dealers supplying Bloomberg with intraday quotes. If you buy the swap, you will pay the ask rate and receive LIBOR. If you sell the swap, you will receive the bid rate and pay LIBOR. The difference between the bid and ask rates is the dealer's

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[10] See International Swaps and Derivatives Association (2000b, p.41).

[11] A midmarket rate is the average of the best bid rate and best ask rate prevailing in the marketplace at a given point in time.

spread. Competition among interest rate swap dealers has driven spreads in the plain-vanilla interest rate market to incredibly small levels—less than 4 basis points on average.

The reasons for entering a fixed-for-floating interest rate swap vary. Because the term structure of interest rates is usually upward sloping, the interest rate on long-term debt is usually higher than short-term debt. Assuming a firm has long-term financing needs, it may want to issue long-term, fixed rate debt so that there is no uncertainty regarding the level of future interest rate payments. On the other hand, a firm may decide to issue floating rate debt because it believes that the average level of interest payments over time will be less than those of a fixed rate loan. A problem with the floating rate alternative, however, is that, while there is good reason to believe that short-term rates will provide lower interest payments on average, it is not guaranteed. An unexpected spike in the short-term rate can have dramatic consequences, particularly when the firm finances much its capital expenditures using internally generated funds. Interest rate swaps are an inexpensive and convenient means of moving back and forth between the two alternative forms of financing. If a firm has fixed rate debt and is willing to incur the risk of floating rate debt in hopes of reducing interest payments, it can enter a fixed-for-floating swap in which it receives fixed rate payment (to offset in whole or in part its payment obligation to its bondholders) and pays floating. If a firm has floating-rate debt and wants to gain the certainty of fixed rate payments, it can enter a fixed-for-floating swap in which it receives floating (to offset in whole or in part its payment obligation to its bondholders) and pays fixed.

The terms of generic interest rate swaps are set such that (1) no money changes hand at inception; (2) no exchange of principal occurs; (3) interest payments are made semiannually and are netted (i.e., the party owing the largest payment pays the difference between the amount he owes and the amount he is supposed to receive); and (4) the floating rate interest payments are based on the six-month LIBOR rate.

The cash flows of a two-year fixed-for-floating swap are summarized in Table 18.3. In the table, the party is assumed to pay fixed and receive floating. The fixed rate is 8%, and is paid semiannually. Note that this implies that 4% of par is paid each period (six months). The floating leg is also paid each six months. The rate is based on the six-month LIBOR rate and is set at the begin-

TABLE 18.3 Hypothetical cash flows of an interest rate swap in which the holder pays fixed and receives floating.

	Time	0	1	2	3	4
Fixed rate leg	Interest		–4.00	–4.00	–4.00	–4.00
	Principal					–100.00
Floating rate leg	Interest		3.50	4.00	4.50	5.00
	Principal					100.00
Net cash flows	Interest		–0.50	0.00	0.50	1.00
	Principal					0.00

ning of each payment period. In the table, the six-month LIBOR rate is 7% at inception, implying that the interest receipt at the end of the first period is already known. The remaining interest receipts are not known at inception. The 4.00, 4.50, and 5.00 receipts are entered only to show the *netting* process, that is, the payments are netted each period, with the party owing the net amount paying the counterparty. Thus in period 1, the fixed rate payer pays –0.50. In period 2, no payment is made, and in periods 3 and 4, the fixed rate payer receives 0.50 and 1.00, respectively. The notional amount of the swap also appears on the terminal date. The net of the notional amounts is zero, implying that the notional amount has no bearing on the valuation of the swap.

ILLUSTRATION 18.1 Transfer risk of floating rate payments.

Suppose that, on July 1, 2004, ABC Company issued $100 million in six-year floating-rate debt at a rate of 100 basis points over six-month LIBOR. Suppose also that over the next year short-term interest rates rise precipitously and ABC becomes concerned that any further increase in short-term rates will take the firm's cash flows to a level that they will not be able to sustain their desired growth rate in investment. What alternatives are available to ABC?

Alternative 1: Take the "Ostrich" strategy. Under this alternative, ABC does nothing. In leaving its short-term interest rate exposure unhedged, the firm is making a bet that short-term rates will stay the same or fall. If they rise, the firm is in trouble.

Alternative 2: Issue fixed rate debt. ABC may have the alternative to retire its floating-rate debt with a fixed rate bond issue. Such an action would lock in interest rate payments and alleviate the firm's short-term interest rate exposure. The main problem with this alternative is that the costs of issuing fixed rate debt may be as high as 250 basis points or more. This means that for every dollar raised, the underwriting firm takes 2.5%.

Alternative 3: Enter a fixed-for-floating swap. Under this scenario, ABC enters a five-year fixed-for-floating swap in which it pays fixed and receives floating (i.e., six-month LIBOR). It checks the current quotes in the OTC market and finds that five-year plain vanilla interest rate swaps are quoted at 4.22-4.26%. Since ABC will pay fixed, the ask rate, 4.26%, is the relevant rate. Assuming it can execute the swap at the prevailing rate, ABC's interest cash flows will appear as follows:

	Payment
Current interest payment	–(LIBOR/2 + 0.50)%
Receive LIBOR	(LIBOR/2)%
Pay fixed	–(4.26/2)% = –2.13%
Net cash flow	2.63%

Note that ABC's floating rate interest payment has not disappeared. Its risk, however, has disappeared since ABC receives LIBOR as part of the swap. ABC's net cash flow each six-month payment period is fixed at 2.63%.

Interest Rate Swap Valuation

As the above description indicates, an interest rate swap is like being long (short) a fixed rate bond and short (long) a floating rate bond. Applying the valuation-by-replication technique, the value of an interest rate swap is the difference between the values of a fixed rate bond and a floating rate bond.

A fixed rate bond is a coupon-bearing bond. It pays a stated rate of interest periodically throughout the bond's life, ending with an interest payment and repayment of the bond's par value or notional amount. To value a fixed rate bond, we take the present value of the promised fixed rate interest payments, that is,

$$PV_{fixed} = \sum_{i=1}^{n} e^{-r_i T_i} FIXED_i + e^{-r_n T_n} NOTIONAL \tag{18.7}$$

where $FIXED_i$ is the amount of the of the fixed rate payment (i.e., the fixed rate times the notional amount, $NOTIONAL$), r_i is the annualized zero-coupon discount rate used to bring the cash flow to the present, T_i is the number of years until the cash flow i occurs, and n is the number of interest payments.

Like a fixed rate bond, a floating rate bond pays interest periodically throughout the bond's life and then repays the principal at the bond's maturity. The difference is that, with a floating rate bond, the periodic interest rate "floats" from period to period. The interest rate is linked to a short-term reference rate such as LIBOR, T-bills, prime, and the Fed Funds rate and is set at the beginning of each payment period (i.e., on the *reset date*). The tenor of the reference rate is typically less than a year. Generic interest rate swaps, for example, are linked to six-month LIBOR.

Conceptually, valuing a floating rate bond may seem more difficult than valuing a fixed rate bond since the amounts of floating rate payments, except for the first, are unknown. To determine the value of a floating rate bond, we must first forecast the expected interest payments, $E(FLOAT_i)$, and then discount the expected interest payments to the present, that is,

$$PV_{floating} = e^{-r_1 T_1} FIXED_1 + \sum_{i=2}^{n} e^{-r_i T_i} E(FLOAT_i) + e^{-r_n T_n} NOTIONAL \tag{18.8}$$

The first payment, $FLOAT_1$, is treated separately to reflect the fact that the amount of the first interest payment is already set. The remaining interest payments are estimated using the forward rates implied by the zero-coupon yield curve.

Fortunately, (18.8) is not the only way to value a floating rate bond. A much simpler approach is possible. To understand this approach, note first that, on a reset date, the six-month LIBOR rate determines the amount of the interest payment in six months. Hence, the value of the floating rate bond in six months is $100(1 + LIBOR)$. Note also that the six-month LIBOR rate is the discount rate we would use to bring a future value occurring six months back to the present. Thus standing on each reset date, the value of a floating rate bond is

$$\frac{100(1 + LIBOR)}{1 + LIBOR} = 100$$

The only time the floating rate leg deviates in value from 100 is in the current period when interest rate payment has been set and the zero-coupon yield curve changes. On the next reset date, the value of the floating-rate bond again reverts to 100. Between reset dates, the value of the floating-rate bond is

$$PV_{floating} = e^{-r_1 T_1}(FLOAT_1 + NOTIONAL) \tag{18.9}$$

With valuation formulas for the fixed rate (18.7) and floating rate (18.9) legs of the interest rate swap, we can now value the swap itself. The value of an interest rate swap from the perspective of someone receiving fixed and paying floating is the difference,

$$V_{swap} = PV_{fixed} - PV_{floating} \tag{18.10}$$

ILLUSTRATION 18.2 Find value of floating rate bond given zero-coupon yield curve.

Suppose that the current zero-coupon yield curve is

$$r_i = 0.04 + 0.01\ln(1 + T_i)$$

Find the value of a five-year floating rate bond with semiannual interest rate payments.

Like any other security, the valuation of the floating rate bond of an interest rate swap is a matter of identifying the amount and the timing of expected future cash flows and then discounting them to the present. To identify expected future cash flows, we use the current zero-coupon yield curve to identify forward rates, and then use forward rates to determine expected interest payments.

Step 1: Find the discount rate (factor) for each cash flow by substituting into the term structure equation. The spot rates and discount factors are shown in the following table. Recall the discount factor is today's price of \$1 received at time T_i, that is, $DF_i = e^{-r_i t_i}$.

Years to Maturity	Spot Rate	Discount Factor	Implied Forward Rate
0.00	4.000%	1.00000	
0.50	4.405%	0.97821	4.405%
1.00	4.693%	0.95415	4.981%
1.50	4.916%	0.92891	5.363%
2.00	5.099%	0.90305	5.646%
2.50	5.253%	0.87694	5.869%
3.00	5.386%	0.85079	6.054%
3.50	5.504%	0.82478	6.211%
4.00	5.609%	0.79901	6.347%
4.50	5.705%	0.77359	6.467%
5.00	5.792%	0.74857	6.575%

Step 2: Find the implied forward rates between adjacent periods. This can be done using the forward rate formula from Chapter 2, that is,

$$f_{i,j} = \frac{r_j T_j - r_i T_i}{T_j - T_i}$$

where $f_{i,j}$ is the *implied forward rate of interest* on a loan beginning at time T_i and ending at time T_j. The implied forward rate on a six-month loan, for example, is

$$f_{0.5,1} = \frac{0.04693(1) - 0.04405(0.5)}{1 - 0.5} = 4.981\%$$

The discount factors in the above table are also inextricably linked to forward rates. The price of a six-month discount bond with a par value of one dollar is 0.97821, and the price of a one-year discount bond is 0.95415. That means that the implied price of a six-month discount bond in six months is 0.95415/.97821 or 0.97540. Its forward rate of return is

$$f_{0.5,1} = \frac{\ln(0.97540)}{0.5} = 4.981\%$$

on an annualized basis.

Years to Maturity	Spot Rate	Discount Factor	Implied Forward Rate	Implied Forward Discount Factor
0.00	4.000%	1.00000		
0.50	4.405%	0.97821	4.405%	0.97821
1.00	4.693%	0.95415	4.981%	0.97540
1.50	4.916%	0.92891	5.363%	0.97354
2.00	5.099%	0.90305	5.646%	0.97217
2.50	5.253%	0.87694	5.869%	0.97108
3.00	5.386%	0.85079	6.054%	0.97018
3.50	5.504%	0.82478	6.211%	0.96942
4.00	5.609%	0.79901	6.347%	0.96876
4.50	5.705%	0.77359	6.467%	0.96818
5.00	5.792%	0.74857	6.575%	0.96766

Step 3: Find the expected floating rate interest payments. Recall that the floating rate used to determine the amount of the floating rate payment is the one prevailing at the beginning of the period. The first floating rate payment is therefore known today and is $100(e^{0.04405(0.5)} - 1) = 2.2272$. The amount of the second floating rate payment is an expected value based on the six-month forward rate starting in six months, that is, $100(e^{0.04981(0.5)} - 1) = 2.5217$. The remaining expected floating rate payments are as shown in the table below.

Years to Maturity	Spot Rate	Discount Factor	Implied Forward Rate	Expected Cash Flow
0.00	4.000%	1.00000		
0.50	4.405%	0.97821	4.405%	2.2272
1.00	4.693%	0.95415	4.981%	2.5217
1.50	4.916%	0.92891	5.363%	2.7176
2.00	5.099%	0.90305	5.646%	2.8630
2.50	5.253%	0.87694	5.869%	2.9782
3.00	5.386%	0.85079	6.054%	3.0733
3.50	5.504%	0.82478	6.211%	3.1541
4.00	5.609%	0.79901	6.347%	3.2244
4.50	5.705%	0.77359	6.467%	3.2865
5.00	5.792%	0.74857	6.575%	103.3421

Step 4: Take the present value of the expected floating rate payments by multiplying each expected payment by the corresponding discount factor. This table summarizes the results:

Years to Maturity	Spot Rate	Discount Factor	Implied Forward Rate	Expected Cash Flow	PV of Expected Cash Flow
0.00	4.000%	1.00000			
0.50	4.405%	0.97821	4.405%	2.2272	2.1786
1.00	4.693%	0.95415	4.981%	2.5217	2.4061
1.50	4.916%	0.92891	5.363%	2.7176	2.5244
2.00	5.099%	0.90305	5.646%	2.8630	2.5855
2.50	5.253%	0.87694	5.869%	2.9782	2.6117
3.00	5.386%	0.85079	6.054%	3.0733	2.6147
3.50	5.504%	0.82478	6.211%	3.1541	2.6014
4.00	5.609%	0.79901	6.347%	3.2244	2.5763
4.50	5.705%	0.77359	6.467%	3.2865	2.5424
5.00	5.792%	0.74857	6.575%	103.3421	77.3590
Total					100.00

Surprisingly, or perhaps not so surprisingly, the present value of the floating rate bond equals 100. The intuition for this result is simple. Since the expected floating rate payment is determined from the forward rates of the zero-coupon yield curve and the zero-coupon yield curve contains the discount factors used to bring the cash flows back to the present, their effects outset each other, making the present value of the loan equal to its par value.

ILLUSTRATION 18.3 Find fixed rate on plain-vanilla interest rate swap given zero-coupon yield curve.

Suppose that the current zero-coupon yield curve is

$$r_i = 0.04 + 0.01\ln(1 + T_i)$$

Find the fixed rate on a five-year, fixed-for-floating, plain-vanilla interest rate swap.

At inception, the value of a swap is 0. Since the present value of the floating rate leg is 100 on a reset date, this means that the fixed rate on a fixed-for-floating swap is that rate that makes the present value of the fixed rate leg equal to 100. This rate cannot be computed directly, and must be determined iteratively using the present value formula (18.7). At a 7% fixed rate, the present value of the fixed rate is too high, as shown below.

Fixed Rate:	**7.0000%**		
Years to Maturity	**Spot Rate**	**Promised Cash Flow**	**PV of Promised Cash Flow**
0.00	4.000%		
0.50	4.405%	3.5000	3.4237
1.00	4.693%	3.5000	3.3395
1.50	4.916%	3.5000	3.2512
2.00	5.099%	3.5000	3.1607
2.50	5.253%	3.5000	3.0693
3.00	5.386%	3.5000	2.9778
3.50	5.504%	3.5000	2.8867
4.00	5.609%	3.5000	2.7965
4.50	5.705%	3.5000	2.7076
5.00	5.792%	103.5000	77.4772
Total			105.0900
Value of swap			5.0900

Since the present value is higher than 100, we must lower the fixed rate. If our next guess is 5.8124%, we will find that the present value of the fixed rate leg is 100. Alternatively, we can use the Microsoft Excel SOLVER function to assist us in our work.

Fixed Rate:	**5.8214%**		
Years to Maturity	**Spot Rate**	**Promised Cash Flow**	**PV of Promised Cash Flow**
0.00	4.000%		
0.50	4.405%	2.9107	2.8473
1.00	4.693%	2.9107	2.7773
1.50	4.916%	2.9107	2.7038
2.00	5.099%	2.9107	2.6285
2.50	5.253%	2.9107	2.5525
3.00	5.386%	2.9107	2.4764
3.50	5.504%	2.9107	2.4007
4.00	5.609%	2.9107	2.3257
4.50	5.705%	2.9107	2.2517
5.00	5.792%	102.9107	77.0361
Total			100.0000
Value of swap			0.0000

The OTC swap dealer will set his bid-ask quotes surrounding this fixed rate. Assuming the bid/ask spread is four basis points, the dealer might quote a bid rate of 5.80% (i.e., the fixed rate the counterparty would receive while paying floating) and an ask rate of 5.84% (i.e., the fixed rate the counterparty would pay while receiving floating).

ILLUSTRATION 18.4 Value of swap between interest payments.

Suppose that we entered the swap in Illustration 18.3, and are receiving fixed at a rate of 5.8214% and paying floating. Two months has elapsed, and the new zero-coupon yield curve is

$$r_i = 0.05 + 0.01\ln(1 + T_i)$$

Compute the current value of the swap.

The current value of the swap, from our perspective, is the present value of the fixed-rate payments (i.e., what we receive) less the present value of the floating rate payments (i.e., what we pay). The first step is to find the discount rate (factor) for each cash flow by substituting into the term structure equation. We next take the present value of the fixed rate payments (equation (18.7)), and, finally, we take the present value of the expected floating rate payments using equation (18.9). The table that follows summarizes the computations. Note that the first payment on the floating rate leg, 2.2272, was set two months earlier.

Fixed Rate Leg			
Fixed Rate:	**5.8214%**		
Years to Maturity	**Spot Rate**	**Promised Cash Flow**	**PV of Promised Cash Flow**
0.00			
0.33	5.288%	2.9107	2.8599
0.83	5.606%	2.9107	2.7779
1.33	5.847%	2.9107	2.6924
1.83	6.041%	2.9107	2.6055
2.33	6.204%	2.9107	2.5184
2.83	6.344%	2.9107	2.4319
3.33	6.466%	2.9107	2.3463
3.83	6.576%	2.9107	2.2622
4.33	6.674%	2.9107	2.1797
4.83	6.764%	102.9107	74.2142
Total			96.8884
Value of swap		-3.5527	

Floating Rate Leg					
Years to Maturity	**Spot Rate**	**Discount Factor**	**Implied Forward Rate**	**Expected Cash Flow**	**PV of Expected Cash Flow**
0.00	5.000%	1.0000			
0.33	5.288%	0.9825	5.288%	2.2272	2.1883
0.83	5.606%	0.9544	5.818%	2.9520	2.8172
1.33	5.847%	0.9250	6.249%	3.1739	2.9359
1.83	6.041%	0.8952	6.559%	3.3340	2.9844
2.33	6.204%	0.8652	6.800%	3.4584	2.9923
2.83	6.344%	0.8355	6.996%	3.5599	2.9742
3.33	6.466%	0.8061	7.161%	3.6454	2.9386
3.83	6.576%	0.7772	7.304%	3.7193	2.8906
4.33	6.674%	0.7489	7.429%	3.7842	2.8338
4.83	6.764%	0.7212	7.540%	103.8421	74.8858
Total					100.4411

Note also that the present value of the floating rate leg was determined using the full set of computations performed in Illustration 18.3. This was unnecessary, since we have already shown that the present value of the floating rate leg is simply the present value of the sum of the next floating rate payment and the notional amount, that is,

$$PV_{floating} = e^{-0.05288(0.333)}(2.2272 + 100) = 100.4411$$

As the table shows, the value of the swap is now –\$3.5527, that is,

$$\begin{aligned}\text{Value of swap} &= PV_{fixed} - PV_{floating}\\ &= 96.8884 - 100.4411\\ &= -3.5527\end{aligned}$$

The fact that the swap has fallen in value from 0 should not be surprising—the duration of the fixed rate leg is higher than the duration of the floating rate leg. Interest rates rose over the past two months, hence the fixed rate leg fell in value by more than the floating rate leg. To unwind the swap, we would have to pay \$3.5527.

Valuation of an Inverse Floater

An *inverse floater* is like a floating rate bond in the sense that its interest payments are based on a reference (i.e., floating) rate.[12] The only difference is that instead of receiving the prevailing floating rate each period, we receive a constant fixed rate less the reference rate (e.g., 10% less six-month LIBOR), that is,

$$\text{Rate on inverse} = \text{Fixed rate} - \text{Reference rate} \tag{18.11}$$

Occasionally the inverse floater will be *leveraged* or *supercharged*, in which case the reference rate is multiplied by a factor λ, where $\lambda > 1$. The rate on a *leveraged inverse floater* is

$$\text{Rate on inverse} = \text{Fixed rate} - \lambda \times \text{Reference rate} \tag{18.12}$$

Occasionally the rate on the inverse will have a cap or a floor too. For ease of exposition, we ignore both of these cases in the valuation and risk measurement discussions below. Since the generic reference rates are either quarterly or semiannual, generic inverse floaters have either quarterly or semiannual interest payments.

In order to value an inverse floater, we must first forecast the expected interest payments $E(INVFLOAT_i)$, and then discount the expected interest payments to the present. The valuation formula is

$$\begin{aligned}PV_{invfloater} &= e^{-r_1 T_1} INVFLOAT_1\\ &\quad + \sum_{i=2}^{n} e^{-r_i T_i} E(INVFLOAT_i) + e^{-r_n T_n} NOTIONAL\end{aligned} \tag{18.13}$$

[12] Inverse floaters first appeared in early 1986, after a period of sustained decreases in interest rates. Investor floaters are well suited for investors who anticipate interest rates to fall. For a detailed discussion of inverse floating rate swap structures, see Das (1994, pp. 428–453).

where , the first payment, is treated separately to reflect the fact that the amount of the first interest payment was set at the beginning of the period and is already known. By definition, the payment on an inverse floater equals a fixed rate less the reference floating rate, that is,

$$INVFLOAT = FIXED - FLOAT \tag{18.14}$$

Thus, given the expected cash flows of a floating rate bond, we can identify the expected cash flows and value of an inverse floater.

ILLUSTRATION 18.5 Value of inverse floater given zero-coupon yield curve.

Suppose that the current zero-coupon yield curve is

$$r_i = 0.04 + 0.01\ln(1 + T_i)$$

Find the value of a five-year, inverse floating rate bond whose payments are 10% less six-month LIBOR.

The steps in the valuation of the inverse floater parallel those used for the floating rate bond in Illustration 18.4.

Step 1: Find the discount rate (factor) for each cash flow by substituting into the term structure equation.

Step 2: Find the implied forward rates between adjacent periods. This can be done using the forward rate formula from Chapter 2, that is,

$$f_{i,j} = \frac{r_j T_j - r_i T_i}{T_j - T_i}$$

where $f_{i,j}$ is the *implied forward rate of interest* on a loan beginning at time T_i and ending at time T_j.

Step 3: Find the expected floating rate interest payments. Recall that the floating rate used to determine the amount of the floating rate payment is the one prevailing at the beginning of the period. The first floating rate payment is therefore known today and is $100(e^{0.04405(0.5)} - 1) = 2.2272$. The first inverse floater payment is, therefore, 10/2 – 2.2272 = 2.7728. The expected of the second floating rate payment is an expected value based on the six-month forward rate starting in six months, that is, $100(e^{0.04981(0.5)} - 1) = 2.5217$. The expected amount of the second inverse floater payment is therefore 10/2 – 2.5217 = 2.4783. The remaining expected floating rate and inverse floating rate payments are as shown in the following table. Note that the last payment of the floating rate loan, 103.3421, is the sum of the interest payment, 3.3421, and principal, 100. Likewise, the last payment of the inverse floater, 101.16579, is the sum of interest, 10/2 – 3.3421 = 1.6579, and principal, 100.

Years to Maturity	Spot Rate	Implied Forward Rate	Expected Forward Discount Factor	Expected Inverse Floater Payment
0.00	4.000%			
0.50	4.405%	4.405%	2.2272	2.7728
1.00	4.693%	4.981%	2.5217	2.4783
1.50	4.916%	5.363%	2.7176	2.2824
2.00	5.099%	5.646%	2.8630	2.1370
2.50	5.253%	5.869%	2.9782	2.0218
3.00	5.386%	6.054%	3.0733	1.9267
3.50	5.504%	6.211%	3.1541	1.8459
4.00	5.609%	6.347%	3.2244	1.7756
4.50	5.705%	6.467%	3.2865	1.7135
5.00	5.792%	6.575%	103.3421	101.6579

Step 4: Take the present value of the expected floating rate payments by discounting each expected payment by the corresponding zero-coupon spot rate. The table below summarizes the results.

Years to Maturity	Spot Rate	Implied Forward Rate	Expected Forward Discount Factor	Expected Inverse Floater Payment	PV of Expected Cash Flows: Floater	PV of Expected Cash Flows: Inverse Floater
0.00	4.000%					
0.50	4.405%	4.405%	2.2272	2.7728	2.1786	2.7124
1.00	4.693%	4.981%	2.5217	2.4783	2.4061	2.3647
1.50	4.916%	5.363%	2.7176	2.2824	2.5244	2.1202
2.00	5.099%	5.646%	2.8630	2.1370	2.5855	1.9298
2.50	5.253%	5.869%	2.9782	2.0218	2.6117	1.7730
3.00	5.386%	6.054%	3.0733	1.9267	2.6147	1.6393
3.50	5.504%	6.211%	3.1541	1.8459	2.6014	1.5224
4.00	5.609%	6.347%	3.2244	1.7756	2.5763	1.4188
4.50	5.705%	6.467%	3.2865	1.7135	2.5424	1.3256
5.00	5.792%	6.575%	103.3421	101.6579	77.3590	76.0983
Total					100.0000	92.9044

As before, the present value of the floating rate bond equals 100. Since the expected floating rate payment is determined from the forward rates of the zero-coupon yield curve and the zero-coupon yield curve contains the discount factors used to bring the cash flows back to the present, their effects outset each other, making the present value of the loan equal to its par value. The present value of the inverse floater's expected cash flows is 92.9044, with no obvious interpretation.

The valuation of an inverse floater can also be addressed in a different manner. Consider the value of the fixed rate bond (18.7) where the fixed rate is one-half the fixed rate in the inverse floater, that is,

$$PV_{fixed/2} = \sum_{i=1}^{n-1} e^{-r_i t_i} FIXED/2 + e^{-r_n t_n}(FIXED/2 + NOTIONAL) \qquad (18.15)$$

Suppose we buy two of the fixed rate bonds valued using (18.7) and sell a floating rate bond valued using (18.9). The portfolio value equals the value of an inverse floater, that is,

$$\begin{aligned}
&2 \times PV_{fixed/2} - PV_{floating} \\
&= \sum_{i=1}^{n-1} e^{-r_i t_i} FIXED + e^{-r_n t_n}(FIXED + 2 \times NOTIONAL) \\
&\quad - e^{-r_1 t_1} FLOAT_1 - \sum_{i=2}^{n} e^{-r_i t_i} E(FLOAT_i) - e^{-r_n t_n} NOTIONAL \\
&= e^{-r_1 t_1}(FIXED - FLOAT_1) + \sum_{i=2}^{n} e^{-r_i t_i}[FIXED - E(FLOAT_i)] \qquad (18.16) \\
&\quad + e^{-r_n t_n} NOTIONAL \\
&= e^{-r_1 t_1}(INVFLOAT_1) + \sum_{i=2}^{n} e^{-r_i t_i}[E(INVFLOAT_i)] + e^{-r_n t_n} NOTIONAL \\
&= PV_{invfloater}
\end{aligned}$$

Since the floating rate loan can be valued succinctly as (18.9) and the fixed rate loan can be valued as (18.7), it is simplest to value the inverse floater as

$$PV_{invfloater} = 2 \times PV_{fixed/2} - PV_{floating} \qquad (18.17)$$

ILLUSTRATION 18.6 Value of inverse floater as difference between two fixed rate bonds and floating rate bond.

Suppose that the current zero-coupon yield curve is

$$r_i = 0.04 + 0.01\ln(1 + T_i)$$

Find the value of a five-year, inverse floating rate bond whose payments are 10% less six-month LIBOR.

Consider the steps in the valuation of the inverse floater in Illustration 18.5, but add the expected cash flows and present value of expected cash flows of the two fixed-rate bonds with 5% (annualized) interest payments.

Years to Maturity	Spot Rate	Implied Forward Rate	Expected Floating Rate Payment	Expected Inverse Floater Payment	Fixed-Rate Payment 5%	PV of Expected Cash Flows		
						Floater	Inverse Floater	Fixed Rate
0.00	4.000%							
0.50	4.405%	4.405%	2.2272	2.7728	5.0000	2.1786	2.7124	4.8911
1.00	4.693%	4.981%	2.5217	2.4783	5.0000	2.4061	2.3647	4.7708
1.50	4.916%	5.363%	2.7176	2.2824	5.0000	2.5244	2.1202	4.6445
2.00	5.099%	5.646%	2.8630	2.1370	5.0000	2.5855	1.9298	4.5153
2.50	5.253%	5.869%	2.9782	2.0218	5.0000	2.6117	1.7730	4.3847
3.00	5.386%	6.054%	3.0733	1.9267	5.0000	2.6147	1.6393	4.2540
3.50	5.504%	6.211%	3.1541	1.8459	5.0000	2.6014	1.5224	4.1239
4.00	5.609%	6.347%	3.2244	1.7756	5.0000	2.5763	1.4188	3.9951
4.50	5.705%	6.467%	3.2865	1.7135	5.0000	2.5424	1.3256	3.8679
5.00	5.792%	6.575%	103.3421	101.6579	205.0000	77.3590	76.0983	153.4572
Total						100.0000	92.9044	192.9044

Note that the difference in the values of the two fixed rate bonds and the floating rate bond equals the value of the inverse floater, that is,

$$192.0044 - 100 = 92.0044$$

The function,

OV_IR_FLOAT_INVERSE(*reset, fixed, npaytr, freq, nxtim, face, term, rate, vd*)

can be used to value an inverse floater. The arguments of the function are: *reset*, the annualized interest rate set at the last reset date (i.e., the rate used at the time of the next payment); *fixed*, the fixed rate; *ncoupr*, the number of coupons remaining; *freq*, the number of coupons per year; *nxtim*, the time to the next coupon payment expressed in years; *face*, the notional amount of the inverse floater; *term*, a vector of times to maturity of zero-coupon rates; *rate*, a vector of zero-coupon rates; and, *vd*, an indicator variable set equal to *v* or *V* to return the value of the inverse floater, or *d* or *D* to return the duration of the inverse floater.

Duration of an Inverse Floater

An unusual feature of an inverse floater is that its value is extremely sensitive to interest rate movements. To compute the duration of an inverse floater, we rearrange (18.16) as

$$2 \times PV_{fixed/2} = PV_{floating} + PV_{invfloat} \tag{18.18}$$

For an additive shift in the zero-coupon yield curve, this means that

$$2 \times D_{fixed/2} = \left(\frac{PV_{floating}}{PV_{fixed/2}}\right) D_{floating} + \left(\frac{PV_{invfloat}}{PV_{fixed/2}}\right) D_{invfloat} \tag{18.19}$$

where D is duration or the percentage change in bond value for a given shift in the yield curve. Rearranging to isolate $D_{invfloat}$, we get

$$D_{invfloat} = \frac{2 \times D_{fixed/2} - \left(\frac{PV_{floating}}{PV_{fixed/2}}\right) D_{floating}}{\frac{PV_{invfloat}}{PV_{fixed/2}}} \quad (18.20)$$

Keeping in mind that the duration of the floating-rate bond is the time until the next interest payment (i.e., a maximum of six months for the six-month LIBOR rate), (18.20) shows that the duration of the inverse floater is about four times the duration of the fixed rate bond (two times the duration of the fixed rate bond in the numerator divided by a quantity approximately equal to 0.5).

ILLUSTRATION 18.7 Find duration of inverse floater.

Suppose that the current zero-coupon yield curve is

$$r_i = 0.04 + 0.01\ln(1 + T_i)$$

Find the duration of a five-year inverse floating-rate bond whose payments are 10% less six-month LIBOR.

First, compute the duration of the two fixed-rate bonds. The individual contributions of the durations of each of the cash flows are summarized below.

Years to Maturity	Spot Rate	Fixed Rate Payment 5%	PV of Fixed Rate Payment	Proportion of Total	Contribution to Total Duration
0.00	4.000%				
0.50	4.405%	5.0000	4.8911	0.02535	0.01268
1.00	4.693%	5.0000	4.7708	0.02473	0.02473
1.50	4.916%	5.0000	4.6445	0.02408	0.03612
2.00	5.099%	5.0000	4.5153	0.02341	0.04681
2.50	5.253%	5.0000	4.3847	0.02273	0.05682
3.00	5.386%	5.0000	4.2540	0.02205	0.06616
3.50	5.504%	5.0000	4.1239	0.02138	0.07482
4.00	5.609%	5.0000	3.9951	0.02071	0.08284
4.50	5.705%	5.0000	3.8679	0.02005	0.09023
5.00	5.792%	205.0000	153.4572	0.79551	3.97755
Total			192.9044	1.0000	4.4688

The duration of the inverse floater is therefore

$$D_{invfloat} = \frac{4.4688 - \left(\frac{100}{192.0044}\right)0.5}{\frac{92.0044}{192.0044}} = 8.7411$$

RISK MANAGEMENT LESSON: ORANGE COUNTY INVESTMENT POOL

The collapse of the Orange County Investment Pool (OCIP) in 1994 has been described as one of the worst "derivatives disasters" in history. Disaster to be sure—the taxpayers of Orange County reportedly lost $1.7 billion, about the same amount as the market capitalization of Bethlehem Steel, a DJIA component, at the time.[13] But was Orange County a derivatives disaster? No, not really. It was an enormous bet on interest rates that went awry.

The key player in the Orange County controversy was Robert L. Citron, Orange Country's Treasurer. As Treasurer, he supervised tax collection and the investment of funds. Like any other municipality, its problem is cash management. Tax revenue is collected a few times during the year, while cash disbursements are made over the entire year. To ensure that cash disbursements are unencumbered, municipalities generally invest tax revenue in highly liquid, short-term money market instruments (or, as noted Chapter 17, reverse repurchase agreements). In this way, funds in the investment pool generate additional revenue but can be withdrawn quickly and without loss as they are needed. But, Citron's strategy was different. In place of investing in short-term instruments, he invested in intermediate-term U.S. Treasuries, agency notes, corporate notes, and certificates of deposit with average maturities of about four years. From a historical standpoint, the yield curve is usually upward sloping. This means that the rates of return on intermediate-term bonds will generally be higher than short-term instruments. If interest rates do not change, a strategy such as Citron's will typically produce returns higher than money-market rates. If interest rates change, however, the situation is less clear. Since the duration of the intermediate-term bonds is higher than the duration of money market instruments, an unexpected increase in rates will cause the prices of the intermediate-term bonds to fall at a much quicker rate than the short-term rates, and vice versa. This is particularly dangerous for a municipality whose cash disbursement needs may require that the intermediate-term bonds be sold at a loss. Thus, at its most basic level, Citron's strategy was speculative. He was placing a bet that interest rates would be stable or fall.

The next twist in Citron's strategy was that he used repo agreements to increase the leverage (and, hence, duration) of the investment portfolio. In June 1990, for example, the investment pool had a leverage ratio of 1.5. A leverage ratio of one implies that the pool has no borrowed funds. A leverage ratio of 1.5 means that Citron had entered repurchase agreements with half the notes in the investment portfolio, and then used the cash proceeds to buy more notes.[14] Although OCIP used term repos, their maturities were six months or less so their effective duration was near zero. Consequently, if the duration of the original asset portfolio was 4, the increased leverage through repo agreements

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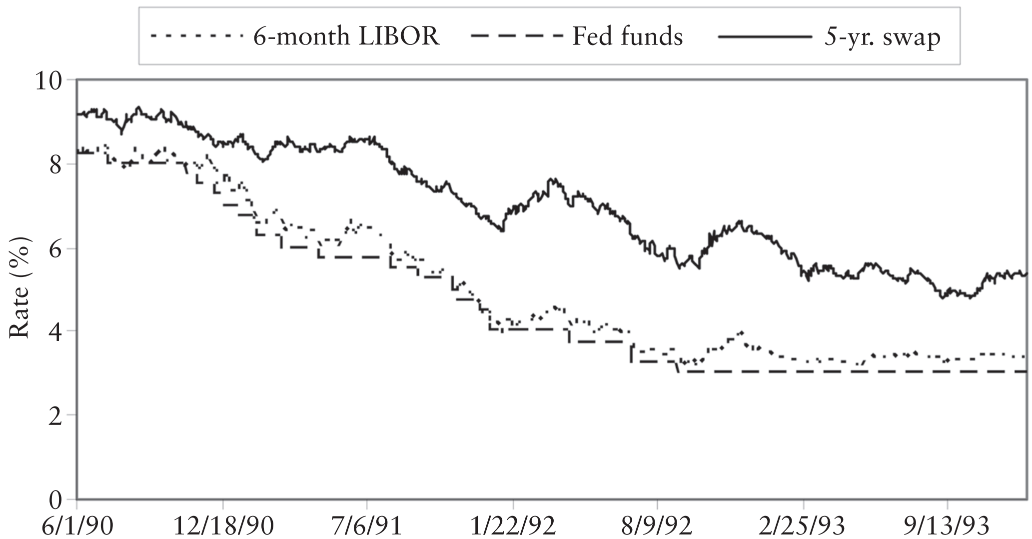
[13] For a very readable and entertaining recount of the Orange County disaster and its chief instigator, County Treasurer Robert L. Citron, see Jorion (1995). Much of the material used in this vignette was drawn from this source. Miller and Ross (1997) argue that, in December 1994, OCIP was neither insolvent nor illiquid and its financial condition did not mandate bankruptcy.

[14] In industry parlance, the interest rate strategy of borrowing short-term and buying long-term is called "riding the yield curve."

increases the fund's duration exposure to 6. In a stable or declining interest rate environment, the strategy could be immensely profitable. And, it was. From the beginning of June 1990 until the end of December 1993, the Federal Reserve lowered the fed funds rate[15] no less than 18 times, taking it from a level of 8.25% to a level of 3% as shown in the figure below. Six-month money market rates (Eurodollar time deposits) fell accordingly, from 8.3125% in June 1990 to 3.4375% in December 1993. At the same time, the yield curve steepened (i.e., the spread between the five-year swap rate and six-month LIBOR widened).



What was Citron's response? Increase leverage, of course. If the bet worked well in the past, why not double up? And, double up he did. By the end of April 1994, the leverage ratio stood at 2.71. Not only had he reversed out of the securities he owned, but he reversed out of the securities he bought with the cash proceeds he received from the original repos. Assuming the intermediate-term bonds in the original portfolio had a duration of 4, the duration of the overall portfolio now stood at a whopping 10.84! In other words, a 100 basis point upward shift in the yield curve would cause the overall portfolio value to fall by nearly 11%.

The table below summarizes the OCIP portfolio as of the end of April 1994. The data were drawn from Jorion (1995, p. 92, Table 10.2). Note that, while the total face value of the securities in the portfolio was $19.86 billion, $12.53 billion of the securities were financed using repo agreements, leaving a net portfolio value of the OCIP of only $7.33 billion. The leverage ratio was $19.86/ $7.33 or 2.71. As noted earlier, the lion's share of the portfolio was invested in intermediate-term Treasury notes, agency notes, corporate notes, and certificates of deposit. The last column contains the average maturity of the securities

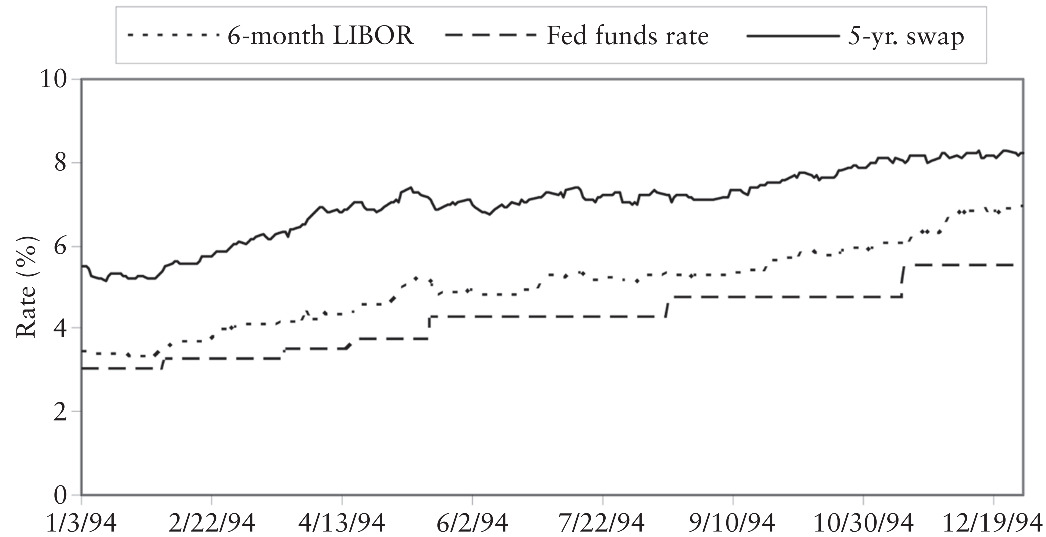
[15] The fed funds rate is set by the Federal Reserve and is a target for the interest rate at which banks lend to each other overnight. While the rates on interbank loans are market-determined, the Fed can influence rates by supplying as much liquidity as there is for demand at the target rate. As the U.S. short-term benchmark, the Fed funds rate influences market interest rates throughout the world.

in each category. For the fixed rate notes, the average maturity of the securities in the category is a rough approximation for the category's duration. For the floating rate agency issues, however, this is not the case. While floating rate agency notes have a duration near zero, the face value of the floating rate notes was less than 10% of the $5.69 billion face value of the "Agency floating rate notes" category. More than two-thirds consisted of about 40 inverse floaters with a weighted average time to maturity of about four years and a weighted average fixed rate of about 11.64% versus six-month LIBOR. Using these average parameters and the zero-coupon yield curve on April 29, 1994, the duration of these inverse floaters was approximately 11.1. With $4 billion in inverse floaters, a one hundred basis point increase in the yield curve would result in a reduction in value of $444 million. Put simply, by the beginning of 1994, OCIP had placed an extraordinarily large bet that interest rates would remain steady or fall.

Asset	Face Value (in millions)	Average Maturity
Treasury notes	582	5
Agency fixed rate notes	8,480	4
Agency floating rate notes	5,693	4
Corporate notes	1,912	4
Mortgage-backed securities	127	10
Certificates of deposit	1,609	4
Mutual funds	421	n/a
Discount notes	686	0
Commerical paper	350	0
Total portfolio value	19,860	
Repos	–12,529	
Net portfolio value	7,331	
Leverage	2.71	

Interest rates in 1994 were anything but steady. The Federal Reserve increased the fed funds rate six times during 1994, as is shown in the figure below. Money-market rates and intermediate-term bond rates also rose. What were the consequences? OCIP suffered extraordinary losses from (1) the decline in value of their leveraged fixed-rate bond position; (2) the decline in value of their inverse floater position; and (3) increased financing costs on the repos.[16] By December 1994, OCIP had reportedly lost $1.7 billion. The positions in the highly leveraged intermediate-term bonds were liquidated, and reinvested in money-market instruments.

[16] In using short-term borrowings to finance the purchase of long-term, fixed rate bonds, one faces risk of the short-term rate rising above the fixed coupon rate.



Clearly, Citron's investment strategy is not difficult to understand. It was a leveraged bet that interest rates would remain steady or fall. The strategy had been profitable in the years prior to 1994 because interest rates fell. When interest rates reversed direction at the beginning of 1994, Citron's fortunes changed for the worse.

Could the situation have been avoided? Absolutely! The investment strategy was entirely inappropriate for a municipality in managing its cash flows. Like controversies discussed in earlier chapters, the culprits are:

1. **Hubris.** Citron's astonishing performance in early years instilled overconfidence, as reflected by the fact that he dramatically increased the leverage of the investment pool through the use of repurchase agreements and inverse floaters. The overconfidence later turned to arrogance when he ignored the warnings of investment banks such as Goldman Sachs and Merrill Lynch about the possible consequences of interest rate advances.
2. **Lack of meaningful supervision.** Nominally, Citron had five elected supervisors. Unfortunately, by most accounts, none of them had a meaningful understanding of OCIP's investment strategy and/or how it was being executed. This situation is particularly egregious for OCIP since no one appeared to question what led to the abnormal performance of the pool. Municipalities aimed at managing cash flows should produce small, safe returns using money market instruments. But Citron's returns were much higher. This should have been the supervisors' red flag. Instead they left him alone to conduct his wizardry.

INTEREST RATE CAPS, FLOORS, AND COLLARS

Interest rate caps and floors are OTC agreements that protect buyers and sellers of floating rate notes against adverse movements in interest rates. A firm with a floating rate loan, for example, faces the risk that its periodic interest payment will jump to a level too high to manage given the firm's current cash flow. By buying an *interest rate cap*, the firm can eliminate its interest rate risk exposure above a specified level. Conversely, an individual holding a floating rate note may want to limit his exposure to rates falling below a certain level. Buying an *interest rate floor* protects the floating rate receiver from such movements.

An *interest rate collar* involves buying an interest rate cap and selling an interest rate floor. The purchase of the cap offers protection from unexpected increases in the floating rate. The sale of the floor subsidizes the cost of the cap through a willingness to forfeit any interest savings if the floating rate falls. Interest rate collars are also marketed as OTC agreements.

There exists a put-call parity relation between the floating rate, a cap, and a floor. If you borrow at a floating rate, buy an interest rate cap with a cap rate of R_X, and sell an interest rate floor with a floor rate of R_X, you have transformed your floating rate loan into a fixed rate loan at R_X.

An important element in valuing caps and floors is contained in the reset mechanics of floating rate loans. Recall that floating rate loans generally have the interest rate set at the beginning of the payment period. Suppose you borrow $100 million for five years at three-month LIBOR. Recall that on such loans, the interest rate is set at the beginning of the period (i.e., on the *reset date*) and interest payment is made at the end. If the current three-month LIBOR rate is 7%, the payment made in three months will be

$$\$100{,}000{,}000 \times (0.075/4) = \$1{,}875{,}000$$

In three months, the interest rate is reset. Suppose, at that time, the three-month LIBOR rate is 8%. The interest payment in six months will be

$$\$100{,}000{,}000 \times (0.08/4) = \$2{,}000{,}000$$

Suppose at the time you borrowed the money, you also bought a 7%, five-year interest rate cap based on three-month LIBOR. By convention, there is no protection on the first interest payment, since its amount is already known. The second payment is protected, however. On the first reset date in three months, the prevailing three-month LIBOR rate (8% in this illustration) is compared with the cap rate, 7%, and the difference in the rates is paid three months later. Thus, although you must make a $2 million interest payment in six months, you will receive a payment of

$$\$100{,}000{,}000 \times [(0.08 - 0.07)/4] = \$250{,}000$$

on the interest rate cap agreement. The net interest payment of $1,750,000 implies an annualized interest rate of 7%, exactly equal to the cap rate.

Valuation of Caps, Floors, and Collars

To value an interest rate cap, we use a portfolio of European-style call options, with each option's expiration corresponding to a reset date of the underlying floating-rate bond. Assuming the forward three-month LIBOR rate at time i, F_i, is lognormally distributed and R_X is the known interest rate cap (i.e., exercise price), the value of the first reset option (called a *caplet*) and R_X is the interest rate cap,

$$c_i = e^{-r_{i+1}t_{i+1}}[F_iN(d_1) - R_XN(d_2)] \tag{18.21}$$

where

$$d_1 = \frac{\ln(F_i/R_X) + 0.5\sigma_i^2 t_i}{\sigma_i\sqrt{t_i}}, \quad d_2 = d_1 - \sigma_i\sqrt{t_i}$$

t_i represents the time until the reset date, and t_{i+1} represents the time until the payment date for the *i*-th reset. Two different times appear in (18.21) because the interest rate is set at the beginning of the reset period while the interest payment is made at the end of the period. Note that the volatility rate, σ_i, is specific to the time to the *i*-th reset date. (We will discuss the term structure of volatility later in this section.) The overall value of the interest rate cap is the sum of the n caplets in the interest rate cap agreement, that is,

$$\text{Cap value} = \sum_{i=1}^{n} c_i \tag{18.22}$$

An interest rate floor agreement can be developed in a similar manner. Since the interest rate floor provides protection against downward movements in the floating rate, each *floorlet* is valued using a put option formula, that is,

$$p_i = e^{-r_{i+1}t_{i+1}}[R_XN(-d_2) - F_iN(-d_1)] \tag{18.23}$$

and the overall value of an interest rate floor is

$$\text{Floor value} = \sum_{i=1}^{n} p_i \tag{18.24}$$

If you buy a cap and sell a floor with the same terms, the value of each combined caplet and floorlet is

$$\begin{aligned} c_i - p_i &= e^{-r_{i+1}t_{i+1}}[F_iN(d_1) - R_XN(d_2)] - e^{-r_{i+1}t_{i+1}}[R_XN(-d_2) - F_iN(-d_1)] \\ &= e^{-r_it_{i+1}}(F_i - R_X) \end{aligned} \tag{18.25}$$

Summing the values across the n payments produces the value of an interest rate swap in which you pay fixed at rate R_X and receive floating.

As noted earlier, the above valuation procedure uses a separate volatility for each period. These volatilities are called *forward forward volatilities* because they are the expected future volatility of the forward rate of interest. That is, each volatility rate is the forward volatility of a one-period forward rate that will exist in the future. It is not surprising, therefore, that some refer to this curve as the *forward volatility curve*. It is more common in practice, however, to see a single volatility used for all of the caplets (floorlets) in the cap (floor) for reporting purposes. These are called *flat volatilities*. If the flat volatilities for caps or floors for a number of maturities are available, you can deduce the spot volatility term structure by using a bootstrapping technique. Bootstrapping is analogous to computing the implied forward rate from the zero-coupon yield curve. If the one-period and two-period flat volatilities are known, we can infer the expected one-period volatility in one period.

ILLUSTRATION 18.8 Value interest rate cap given zero-coupon yield curve and a flat volatility rate curve.

Suppose that the current zero-coupon yield curve is

$$r_i = 0.05 + 0.01\ln(1 + T_i)$$

and that the flat volatility rate on a one-year cap is 30%. Compute the value of a one-year, 6% interest rate cap where the underlying floating rate loan has quarterly payments. Assume that the notional amount of the loan is \$100,000.

The first step is to generate the zero-coupon yield curve and deduce the implied forward rates. The spot rates in the table below are computed directly from the zero-coupon yield curve given above. The continuously componded forward rates are computed in the usual fashion, that is,

$$f_{i,j} = \frac{r_j T_j - r_i T_i}{T_j - T_i}$$

where $f_{i,j}$ is the implied forward rate of interest on a loan beginning at time T_i and ending at time T_j. To convert the continuously compounded forward rate to a quarterly-compounded rate (i.e., the standard manner in which Eurodollar rates are quoted), we use

$$f^Q_{i,j} = \frac{e^{f_{i,j}(T_j - T_i)} - 1}{T_j - T_i}$$

The three-month forward rate in six months, for example, is

$$f^Q_{i,j} = \frac{e^{0.05588(0.5 - 0.25)} - 1}{0.5 - 0.25} = 0.05627$$

Years to Maturity	Spot Rate	Discount Factor	Implied Forward Rate	
			Continuous	Quarterly
0.00	5.000%	1.00000		
0.25	5.223%	0.98703	5.223%	5.257%
0.50	5.405%	0.97333	5.588%	5.627%
0.75	5.560%	0.95916	5.868%	5.911%
1.00	5.693%	0.94466	6.094%	6.140%

The next step is to value each of the caplets with the cap. Since the interest rate payment in three months has already been set, there is no caplet corresponding to the interest rate payment in three months. The value of the caplet corresponding to the payment in six months is

$$c_i = 100{,}000e^{-0.05405(0.5)}[(0.05257/4)N(d_1) - (0.06/4)N(d_2)] = 21.285$$

where

$$d_1 = \frac{\ln(0.05257/0.06) + 0.5(0.30)^2(0.25)}{0.30\sqrt{0.25}} \quad \text{and} \quad d_2 = d_1 - 0.30\sqrt{0.25}$$

Note that the forward rate and cap rate have been divided by four because the rates are annualized and the payments are quarterly. The value of each caplet is multiplied by 100,000 to account for the notional amount of the floating rate loan. For convenience, the value of each caplet can be computed using the function

OV_TS_VALUE_CAPLET(*f*,*rx*,*t1*,*t2*,*r2*,*v1*)

where *f* is the forward rate, *rx* is the cap rate, *t1* is the time until the reset date, *t2* is the time until the reset date payment, and *v1* is the volatility rate corresponding to time *t1*. For the caplet whose payment occurs in six months, the function produces a numerical value of

OV_TS_VALUE_CAPLET(0.05257/4,0.06/4,0.25,0.5,0.05405,0.30) = 0.00021285

Multiplying by the notional amount of the loan, the caplet value is 21.285.

The remaining caplets are computed in a similar fashion. The value of the cap is 234.675, as is shown in this table:

Years to Maturity	Spot Rate	Discount Factor	Implied Forward Rate: Continuous	Implied Forward Rate: Quarterly	Value of Caplet
0.00	5.000%	1.00000			
0.25	5.223%	0.98703	5.223%	5.257%	
0.50	5.405%	0.97333	5.588%	5.627%	21.285
0.75	5.560%	0.95916	5.868%	5.911%	78.359
1.00	5.693%	0.94466	6.094%	6.140%	135.121
				Cap value	234.765

VALUATION OF SWAPTIONS

A *swaption* is an option on an interest rate swap. It gives its holder the right to enter into a certain interest rate swap at a certain time in the future. A firm may know, for example, that in six months it will need to enter into a five-year floating-rate loan agreement and will want to swap the floating rate interest payments for fixed rate interest payments. By buying a swaption, the firm receives the right to receive six-month LIBOR and pay a fixed certain rate for a five-year period beginning in six months. The specified fixed rate of the swaption is its exercise price. If the rate on a five-year fixed versus floating interest rate swap is less than the exercise price in six months, the firm will exercise the swaption. If it is greater, the firm will choose not to exercise and will enter a swap in the marketplace. Because the firm has the right, but not the obligation, to enter the swap underlying the swaption, it must pay for the privilege. Naturally, the firm also has the alternative of entering a *forward* or *deferred swap* with no up-front cost. Like all forward contracts, however, the firm is obligated to enter into the swap agreement whether or not the terms are favorable relative to the then-prevailing market rates.

An interest rate swap is an agreement to exchange a fixed rate bond for a floating rate bond. At the start of the swap, the value of a floating rate bond always equals the principal amount of the swap. A swaption can therefore be regarded as an option to exchange a fixed rate bond for the principal amount of the swap. If a swaption gives the holder the right to pay fixed and receive floating, it is a put option on the fixed rate bond with an exercise price equal to the principal. If a swaption gives the holder the right to pay floating and receive fixed, it is a call option on the fixed rate bond with an exercise price equal to the principal.

Valuation of Swaptions

Like in the valuation of caps and floors, the valuation of a swaption assumes that the underlying forward (swap) rate is distributed log-normally at the option's expiration. The volatility of the forward rate, therefore, is the volatility of a forward fixed rate on a fixed-for-floating swap. Suppose that at the swaption's expiration, the rate on an n-year swap is R. By comparing the cash flows on a swap where the fixed rate is R to the cash flows on a swap where the fixed rate is R_X, we see that the payoff from the swaption consists of a series of cash flows equal to

$$\frac{L}{m}\max(R - R_X, 0) \tag{18.26}$$

where L is the principal amount of the swap, and both and are expressed with a compounding frequency of m times per year.

The cash flows are received m times per year for the n years of the life of the swap. Suppose that the payment dates are $t_1, t_2, \ldots, t_m$ measured in years. Each cash flow is the payoff from a call on R with strike price R_X. In other words, you do not need a separate option value for each cash flow as you did for caps and floors. One will suffice. The value of the cash flow at time t_i (where $t_i = T + i/m$) is

$$\frac{L}{m}e^{-r_i t_i}[FN(d_1) - R_X N(d_2)] \tag{18.27}$$

where

$$d_1 = \frac{\ln(F/R_X) + 0.5\sigma^2 T}{\sigma_i\sqrt{T}}, \quad d_2 = d_1 - \sigma_i\sqrt{t_i}$$

is the forward rate on an n-year swap that begins at time T, and r_i is the continuously compounded zero-coupon interest rate for maturity t_i. The *swaption value* is therefore

$$\sum_{i=1}^{mn} \frac{L}{m} e^{-r_i t_i} [FN(d_1) - R_X N(d_2)] \tag{18.28}$$

Some of you will recognize that this formula is the present value of an annuity, that is,

$$\frac{L}{m} [FN(d_1) - R_X N(d_2)] \sum_{i=1}^{mn} e^{-r_i t_i} \tag{18.29}$$

The value of a put option is

$$\frac{L}{m} e^{-r_i t_i} [R_X N(-d_2) - FN(-d_1)] \tag{18.30}$$

Finally, it is worth noting that both caps/floors and swaptions are quoted in terms of the Black (1976) model in the marketplace even though it is theoretically inconsistent to do so. The cap/floor market uses the short-term LIBOR rate as the underlying source of uncertainty, while the swaptions market uses longer-term forward rates. Since forward swap rates are nearly linear in the individual forward rates, the log-normality assumption implicit in the Black model cannot hold simultaneously for both individual forward rates and forward swap rates (i.e., a linear combination of log-normal variates is not log-normal). Among other things, this means that direct comparisons between quoted implied volatilities for caps/floors and swaptions are improper. A general, all-encompassing (albeit more computationally intensive) framework for valuing interest rate products is provided in the next chapter.

ILLUSTRATION 18.9 Value swaption.

Suppose that the zero-coupon yield curve based on LIBOR is flat at 4% compounded continuously. Compute the value of a three-year option on a five-year swap assuming the swaption gives the holder the right to receive 4.2% fixed. Assume payments are made semiannually and principal is 100. Assume also that the volatility of the forward rate on five-year swaps in three years is 30%.

The right to receive fixed is a put option. You will exercise only when the fixed rate on the five-year swap in three years is below 4.2%.

The put option swaption formula is

$$\text{Put on swap} = \frac{L}{m} [R_X N(-d_2) - FN(-d_1)] \sum_{i=1}^{mn} e^{-r_i t_i}$$

The sum of the discount factors is

Years to Maturity	Continuous Rate	Discount Factor	Sum of Discount Factors
0.00	4.000%	1.000000	
0.50	4.000%	0.980199	
1.00	4.000%	0.960789	
1.50	4.000%	0.941765	
2.00	4.000%	0.923116	
2.50	4.000%	0.904837	
3.00	4.000%	0.886920	
3.50	4.000%	0.869358	0.869358
4.00	4.000%	0.852144	0.852144
4.50	4.000%	0.835270	0.835270
5.00	4.000%	0.818731	0.818731
5.50	4.000%	0.802519	0.802519
6.00	4.000%	0.786628	0.786628
6.50	4.000%	0.771052	0.771052
7.00	4.000%	0.755784	0.755784
7.50	4.000%	0.740818	0.740818
8.00	4.000%	0.726149	0.726149
Total			7.958452

The value of the put in the squared brackets is

$$[R_X N(-d_2) - FN(-d_1)] = 0.042N(0.3537) - 0.040N(-0.1659) = 0.009441$$

The value of the swaption is

$$\text{Put on swap} = \frac{100}{2}[0.009441]7.958452 = 3.7567$$

SUMMARY

This chapter deals with OTC interest rate products which have multiple cash flows through time. A critical component in accurately valuing such derivative contracts is knowing how to measure the zero-coupon yield curve. The first section describes some commonly used data sources and estimation procedures. With the zero-coupon curve in hand, we then focus on the valuation of fixed-for-floating interest rate swaps and how they are used for risk management purposes. We then turn to the valuation of interest rate caps, collars, and floors, as well as swaptions.

REFERENCES AND SUGGESTED READINGS

Black, Fischer. 1976. The pricing of commodity contracts. *Journal of Financial Economics* 3 (March): 167–179.

Das, Satyajit. 1994. *Swap and Derivative Financing: The Global Reference to Products, Pricing, Applications and Markets*, revised ed. Chicago, IL: Probus.
International Swaps and Derivatives Association. 2000a. *2000 ISDA Definitions*. New York: International Swaps and Derivatives Association.
International Swaps and Derivatives Association. 2000b. *Annex to the 2000 ISDA Definitions*, June 2000 version. New York: International Swaps and Derivatives Association.
Jorion, Philippe. 1995. *Big Bets Gone Bad*. San Diego, CA: Academic Press.
Miller, Merton H., and David J. Ross. 1997. The Orange County bankruptcy and its aftermath: Some new evidence. *Journal of Derivatives* (Summer): 51–60.
Tuckman, Bruce. 2002. *Fixed Income Securities*. 2nd ed. Hoboken, NJ: John Wiley & Sons.