CHAPTER 2

Assumptions and Interest Rate Mechanics

This book deals with risk management using derivatives. Effective risk management, however, requires accurate risk measurement, and accurate risk measurement requires a thorough understanding of valuation. The purpose of this chapter and the next is to review the fundamental principles of security valuation. This chapter focuses on the key assumptions that underlie security valuation models and reviews the use of interest rate mechanics in moving expected future cash flows through time. The next chapter focuses on estimating appropriate discount rates for securities given their risk characteristics.

The outline of this chapter is as follows. The first section presents the set of assumptions that underlie our valuation framework. The second section deals with the interest rate mechanics that allow us to move cash flows through time. The third and fourth sections then apply the assumptions and interest rate mechanics to value fixed income securities—discount bonds and coupon bonds. The fifth section focuses on the relation between interest rates and term to maturity as well as the meaning and computation of forward rates of interest. The sixth section describes common stock valuation. The chapter concludes with a summary.

UNDERLYING ASSUMPTIONS

Building valuation models requires making assumptions. Two assumptions that lay the foundation for security valuation are the absence of costless arbitrage opportunities and frictionless markets. The first assumption is critical; the second is made largely for expositional convenience.

Absence of Costless Arbitrage Opportunities

The absence of costless arbitrage opportunities is driven by a basic tenet of human behavior—individuals prefer more wealth to less, holding other factors constant. "Greed is good!"[1] If two perfect substitutes are traded in the marketplace and they do not have the same price, someone will immediately step in to earn a risk-free profit by simultaneously buying the cheaper asset and selling the more expensive one. Because the asset is both bought and sold simultaneously (albeit in different markets), there is no risk. This is the single key element of an *arbitrage* strategy.[2] Because this particular arbitrage involves no cash outlay, it is a *costless arbitrage*. The person enacting the strategy is called an *arbitrageur.* Because the prices of perfect substitutes must be the same in equilibrium, this principle is also known as the *law of one price*.

Arbitrageurs are at work in all markets where perfect substitutes are traded simultaneously. The shares of IBM, for example, trade on many exchanges in the U.S., not to mention other countries worldwide. Suppose that we see that IBM's stock has a bid price of $120.75 per share on the New York Stock Exchange (NYSE) and an ask price of $120.25 per share on the Pacific Coast Exchange (PCE). We can earn a costless arbitrage profit of $0.50 per share by simultaneously selling IBM on the NYSE and buying it on the PCE. Do not expect to find such opportunities, however. Market makers on the various exchanges continuously monitor markets for such anomalies, and act immediately upon finding any pricing distortion that exceeds trading costs.

Frictionless Markets

Frictionless markets is an assumption made more for convenience than necessity. Invoking it permits sharper focus on the economics of the situation at hand, absent the effects of market idiosyncrasies. Once the economic intuition is developed, the effects of trading costs, taxes, divergent borrowing and lending rates, and the like can be added straightforwardly. For now, however, we wipe the slate clean.

The assumption of frictionless markets requires:

- No trading costs.
- No taxes.
- Unlimited borrowing and lending at the risk-free rate of interest.
- Freedom to sell (short) with full use of any proceeds.
- Can trade at any time.

No Trading Costs Trading costs are costs associated with executing a transaction. These include (1) commissions paid to brokers as well as (2) bid/ask spreads and (3) market impact costs paid to market makers. The effects of trading costs can modeled quite easily. Take, for example, the IBM arbitrage illustration provided earlier in the chapter. Recall that we implicitly incorporated the effect of the bid/

[1] This is from a speech by Gordon Gekko to Teldar Paper Shareholders in the 1987 movie, *Wall Street*, directed by Oliver Stone. See www.americanrhetoric.com/Movie Speeches/moviespeechwallstreet.html.

[2] The term, arbitrage, is frequently misapplied. *Risk arbitrage*, for example, refers to a trading strategy in which the shares of a firm rumored to be on the verge of being acquired are purchased and the shares of the acquiring firm are simultaneously purchased. Since the merger may or may not take place, this activity is *not* arbitrage.

ask spread by comparing the bid price (the price at which we can sell immediately) on the NYSE with the ask price (i.e., the price at which we can buy immediately) on the PCE. Suppose that, in addition to the market maker's spread, our broker charges a commission rate of $0.10 per share. We can still earn a costless arbitrage profit of $0.40. Beyond commissions and spreads, we may face market impact costs if you attempt to trade in large quantities. Since exchanges are obliged to have a minimum market depth at the prevailing market quotes, some amount of profitable arbitrage can be earned. Going beyond that posted levels of depth requires estimating the price elasticity of the stock. Thus, in general, we can account for the effects of trading costs in a logical and coherent fashion because they are known or can be estimated reasonably precisely.

No Taxes Taxes affects valuation in two ways: first, it reduces the amount of the gain (loss), and, second, it may affect the gain (loss) differentially depending upon whether it comes in the form of ordinary income or capital gain. In some models, the first consideration is unimportant. In the IBM arbitrage illustration, the after-trading cost gain was $0.40. Assuming the marginal tax rate is less than 100%, the arbitrage opportunity still exists. The second consideration can have more far reaching consequences, however. Consider two identical firms, one that pays a generous cash dividend each quarter (and raises capital for new investment by issuing new securities) and another that pays no dividends (and uses the cash for new investment). If our long-term capital gains tax rate is less than our ordinary income tax rate, we will prefer to hold the shares of the second firm, holding other factors constant. Taxes, per se, do not make the security valuation problem more complicated, just more tedious. Because the marginal tax rates on the different forms of income are known or can be estimated, incorporating them directly in the valuation problem is straightforward.

Unlimited Borrowing and Lending at the Risk-Free Interest Rate This assumption has two important facets. First, it says that the borrowing and lending rates are equal. Obviously, this is not the case. A bank has a margin between the rate it pays on demand deposits and the rate it charges on short-term loans. Second, it assumes that everyone is equally creditworthy. Borrowing and lending rates vary by customer, with the largest and most secure customers receiving the most favorable rates (i.e., the lowest margin). Because rates are known, accountingfor the effects of divergent borrowing and lending rates within the valuation framework, like trading costs and taxes, is manageable.

Freedom to Sell (Short) with Full Use of Any Proceeds For large institutions, short selling of securities with full use of proceeds is common. Suppose, for example, that we believe that the price of IBM will fall from, say, $120 to $100 over the next month. If we short sell IBM, we will see $120 in cash appear in our account and will have a liability of one share of IBM. Since we have access to the cash, we can invest it immediately and earn interest while our short sale position is in place. When (or if) the price drops to $100, as we predicted, we buy a share of IBM to cover our short position. Our net gain is $120 plus interest less $100. For retail customers, short sales are costly in the sense that the broker may not pay interest on the cash generated from the short sale. Also, for securities in short supply, short sales may not be possible. Under the frictionless market assumption, we have full use of proceeds.

Can Trade at Any Time In order to execute arbitrage, the markets for the perfect substitutes must be open at the same time. Suppose that in late morning London time we see that IBM's shares are quoted at \$121.00 (bid) and \$121.25 (ask) on the London Stock Exchange, while IBM's shares closed at \$120.75 (bid) and \$120.875 (ask) at the NYSE on the previous day. Does that mean a costless arbitrage opportunity is available? Obviously not! The NYSE is not open, so we cannot simultaneously sell in London and buy in New York. Under the frictionless markets assumption, the markets for all securities are open all of the time.

INTEREST RATE MECHANICS

The next step in preparing to value securities is to review interest rate mechanics, that is, how to move expected cash flows through time. Throughout this book, we use *continuously compounded* interest rates. Continuous rates are realistic, convenient, and consistent with the practice of dynamic risk management. Other types of interest rates are mentioned periodically in the discussion, but only when it is necessary to unravel the mystery of the pricing conventions used in a particular market.

Continuously Compounded Interest Rates

Interest rates follow a number of conventions. The first and, perhaps, simplest convention is that interest rates are quoted on an *annualized* basis. This is done to facilitate comparisons across different investment alternatives. If one investment promises a 40% return over five years and another promises a 23% return over three years, it is not immediately obvious which investment we prefer. On the other hand, if we are told that the first investment promises 6.96% annually and the second investment 7.14% annually, the choice is obvious. We are comparing apples with apples.

A second convention is that rates are usually quoted as *nominal rates*. If a bank advertises that it pays 6% *compounded* semiannually, they nominally pay 6% per year (recall the first convention). What they actually pay is, however, 3% each 6 months (i.e., the nominal interest rate divided by the number of compounding intervals in a year). Because interest on interest is earned in the second 6-month period, the effective annual interest rate is $(1 + 0.06/2)^2 = 6.09\%$. In general, given a nominal rate of interest r and m compounding intervals a year, the *effective* interest rate is determined by

$$\text{Effective rate} = (1 + r/m)^m - 1 \tag{2.1}$$

Holding the nominal interest rate constant, the effective interest rate rises with the number of compounding intervals. As m approaches infinity, the effective interest rate becomes

$$\text{Effective rate} = e^r - 1 \tag{2.2}$$

In (2.2), r is referred to as a *continuously compounded* nominal rate of interest.

On first appearance, continuous interest rates may seem unrealistic, but just the opposite is true. Suppose we are interested in modeling the growth of a tree. A tree does not grow by a discrete amount each few months throughout the year. It grows continuously. If the current height of the tree is 50 feet and it grows at a rate of 5% a year, the height of the tree in 6 months will be $50e^{0.05(0.5)} = 51.266$ feet.

The prices of financial instruments grow in exactly the same way. For risky securities such as stocks, prices evolve through time as new information arrives in the marketplace. Growth is continuous in the sense that the movement of the stock price is smooth through the day, however, the rate of movement changes. For risk-free securities, the rate of price movement is constant. Assuming a zero-coupon bond grows at a rate of r percent annually, an investment of B will have a value of F at time T, where F is given by the formula,

$$F = Be^{rT} \tag{2.3}$$

If the growth rate is 6% and the bond's price is \$100, its price will be $F = 100e^{0.06(3/12)}$ = 101.511 in three months, $F = 100e^{0.06(6/12)} = 103.045$ in six months, and so on.

DISCOUNT BONDS

With the continuously compounded interest rate mechanics in hand, we now turn to the valuation of bonds or so-called *fixed income securities*. Bonds are of two types—*zero-coupon* (or *discount bonds*) and *coupon-bearing bonds*. This section focuses on the discount bonds. Coupon-bearing bonds follow in the next. We begin by describing discount bond valuation, and then use the valuation formula as a means of measuring interest rate risk exposure. We follow with a description of the discount instruments issued by the U.S. Treasury.

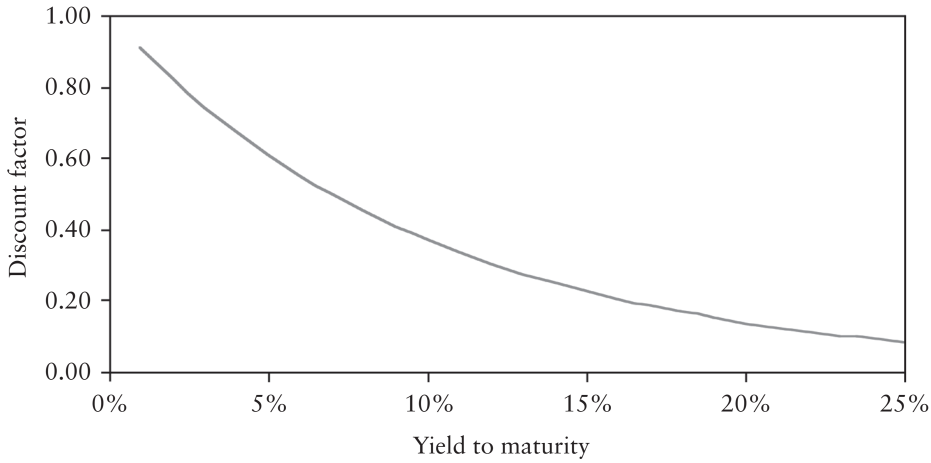
Valuation

A *discount bond* or *zero-coupon bond* is a debt security with a *single* future cash payment, F. F is usually called the *par amount* or *face value* of the bond. If the discount bond has an annualized yield of r percent and a time to maturity of T years, it is

$$B = Fe^{-rT} \tag{2.4}$$

The term, e^{-rT}, is called a *discount factor*. It is the current price of \$1 received at time T. Figure 2.1 shows the discount factors as a function of yield to maturity. Note that the yield and the discount factor are inversely related. The higher the yield, the lower the discount factor. Note also that the function is convex. As yield increases, the bond's value decreases at a decreasing rate. Rearranging (2.4), we can compute the rate of return on a discount bond given its current price, par amount, and term to maturity, that is,

FIGURE 2.1 Discount factor as a function of yield to maturity.



$$r = \frac{\ln(F/B)}{T} \tag{2.5}$$

ILLUSTRATION 2.1 Compute implied yield of discount bond.

In the early 1980s, a number of banks marketed discount bonds to retail customers as a long-term savings vehicle for future expenditures such as their children's college tuition. Interest rates were so high at the time that it was not uncommon to see advertisements saying that a four cent investment today will provide one dollar in 25 years. What is the implied annualized rate of return on this investment?

The annualized rate of return or yield on this investment is

$$r = \frac{\ln(1/0.04)}{25} = 12.876\%$$

This value may be computed using the OPTVAL function,

OV_IR_DISCB_YIELD(*price*, *face*, *term*)

where *price* is the current price of the bond, *face* is its face value, and *term* is its term to maturity. Using the parameters of the problem,

B4 =OV_IR_DISCB_YIELD($B$1,$B$2,$B$3)

	A	B	C	D	E
1	Present value	0.040			
2	Future value	1.000			
3	Years to maturity	25			
4	Implied interest rate	12.876%			
5					

Risk Measurement

In holding a fixed income security such as a discount bond, we are often concerned with knowing what will happen to the value of our bond if interest rates change. Such risk measures are easy to develop once we know how to value the bond. One approach is to simply change the yield in the valuation formula (2.4) from its current level to see what happens to bond value. Indeed, this was the procedure used to generate Figure 2.1. Unfortunately, different bonds react to changes in interest rates in very different ways. To isolate the essential interest rate risk characteristics of a bond, we approximate the shape of the bond valuation function using a polynomial function. Specifically, we expand the bond valuation function (2.4) into a Taylor series about the current yield r_0,[3] that is,

$$dB = \frac{dB}{dr}(r - r_0) + \frac{1}{2}\frac{d^2B}{dr^2}(r - r_0)^2 + \frac{1}{6}\frac{d^3B}{dr^3}(r - r_0)^3 + \cdots \tag{2.6}$$

What (2.6) says is that the change in the bond valuation function (2.4) for a given change in yield equals a polynomial function with an infinite number of terms. As we proceed through the terms on the right-hand side of (2.6), however, they become progressively smaller in size. Interest rate risk management usually involves only the first or, perhaps, the first and the second terms of the series. Higher-order terms are usually ignored.

Let us begin with a first-order approximation. It goes by a variety of names including *DV01* and *duration*. Ignoring second- and higher-order terms on the right-hand side of (2.6), the approximate change in bond value for a given change in yield in given by

$$dB \approx \frac{dB}{dr}(r - r_0) \tag{2.7}$$

where the derivative dB/dr is determined from the valuation equation (2.4), that is,

$$\frac{dB}{dr} = -TFe^{-rT} \tag{2.8}$$

DV01 The acronym, *DV01*, stands for the dollar value of one basis point (i.e., 0.01 of 1%). To create the appropriate formula for *DV01*, we substitute (2.8) into (2.7) and replace $r - r_0$ with 0.0001 and get

$$dB \approx -TFe^{-rT}(0.0001) \equiv DV01 \tag{2.9}$$

[3] A Taylor series expansion can be used to approximate any smooth nonlinear function such as the bond valuation equation. For more details regarding this application, see Appendix 2A.

Duration *Duration* is the percent change in bond value for a given change in yield,[4] and is also commonly used as a measure of interest rate risk exposure. To understand its origin, divide (2.7) by the current bond price, that is,

$$dB/B \approx \frac{dB/B}{dr}(r - r_0) \tag{2.10}$$

Since duration is defined as minus the percent change in bond price with respect to a change in yield,

$$\text{DUR} \equiv -\frac{dB/B}{dr}$$

we have

$$\text{DUR} \equiv -\frac{dB/B}{dr} = -\frac{dB/dr}{B} = -\frac{-TFe^{-rT}}{Fe^{-rT}} = T \tag{2.11}$$

The duration of a discount bond equals the negative of its years to maturity, and, given the value of duration, the percent change in a discount bond value for a given change in interest rates can be approximated using

$$dB/B \approx -T(r - r_0) \tag{2.12}$$

In other words, if a bond has T years to maturity, a one basis point increase in the bond's yield will cause its value to fall approximately $0.01 \times T$%. Note that the *DV01* measure (2.9) gives the same result after we divide through by the bond value.

ILLUSTRATION 2.2 Use duration to approximate discount bond price change.

Suppose that 25 years ago you bought \$4,000 worth of the discount bonds in Illustration 2.1. What would have happened to the value of the bonds if interest rates would have immediately jumped by 100 basis points? Compute the actual change in price using the bond formula (2.4), and then the approximate change using duration (2.10).

At a yield of 12.876%, the value of your investment at inception was \$4,000. If the interest rate jumps to 13.876%, your investment value will fall to

$$B = 100{,}000e^{-0.13876(25)} = 3{,}115.20$$

This can be verified using the OPTVAL Library function

OV_IR_DISCB(*face*, *rate*, *term*, *vdc*)

[4] The concept of duration was first introduced in Macaulay (1938). Other treatments are provided in Reddington (1952) and Samuelson (1945).

where *face* is the face value of the discount bond, *rate* and *term* are its yield and term to maturity, respectively. The indicator variable *vdc* instructs the function to return the bond's value ("*v*" or "*V*"), duration ("*d*" or "*D*"), or convexity ("*c*" or "*C*"). The bond's value is illustrated below. With a 100 basis point increase in the interest rate, the bond value falls by $884.20.

B10 =OV_IR_DISCB($B$2,$B$9,$B$3,"V")

	A	B	C
1	Original bond price	4,000	
2	Face amount	100,000	
3	Years to maturity	25	
4	Original interest rate	12.876%	
5	Duration	25	
6	Convexity	625	
7			
8	Change in interest rate	1.00%	
9	New interest rate	13.876%	
10	New bond value	3,115.20	
11	Actual change in bond value	-884.80	

The duration-based approximation is given by (2.10). Multiplying (2.10) by the bond price provides an estimate of the change in bond value. Since the duration of your bond is 25, an increase of 100 basis points implies that the value of your bond will fall by approximately 25% or $1,000, that is,

$$dB \approx -B \times T \times (r - r_0) = -4{,}000 \times 25 \times 0.01 = -1{,}000$$

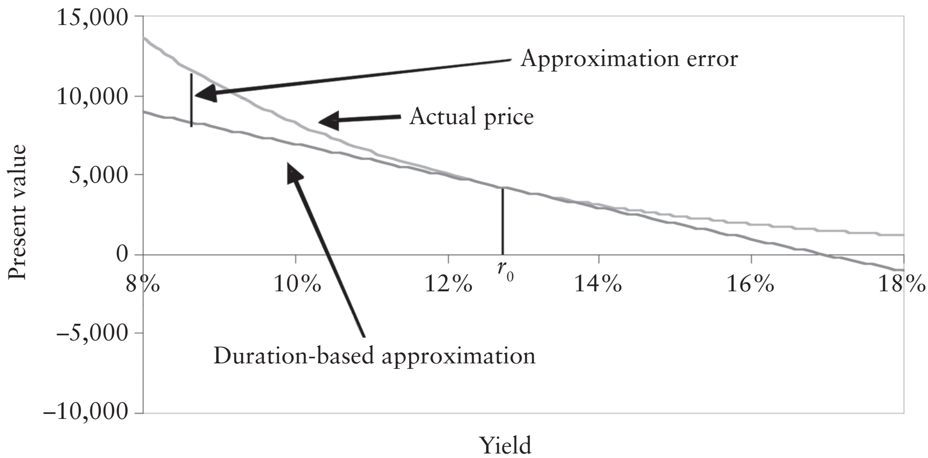
The price discrepancy arises from the fact that the bond valuation function is convex. (See Figure 2.1.) First-order approximations such as duration are accurate for only small changes in yield. As yield changes become large, the degree of error using the duration approximation becomes large.

Convexity DV01 and duration first-order approximations of the bond valuation function that are based on the slope of a straight line that is tangent to the bond valuation function at the current yield, r_0, as shown is Figure 2.2. For small changes in yield, a first-order approximation will be reasonably accurate, however, the approximation error grows large with the size of the yield change. To improve the degree of accuracy in the approximation, we can also incorporate the second-order term of the Taylor series expansion (2.6). Using percent changes, the approximation is now

$$dB/B \approx \left(\frac{dB/B}{dr}\right)(r - r_0) + \frac{1}{2}\left(\frac{d^2B/B}{dr^2}\right)(r - r_0)^2 \tag{2.13}$$

The second term in parentheses on the right-hand side of (2.13) is called *convexity*. Since the second derivative of the bond valuation function is

FIGURE 2.2 Slope of bond valuation formula.



$$\frac{d^2B}{dr^2} = T^2Fe^{-rT} \tag{2.14}$$

the definition of convexity of a discount bond is

$$\text{CVX} = \frac{d^2B/B}{dr^2} = \frac{d^2B/dr^2}{B} = \frac{T^2Fe^{-rT}}{Fe^{-rT}} = T^2 \tag{2.15}$$

ILLUSTRATION 2.3 Use duration and convexity to approximate discount bond price change.

Reconsider Illustration 2.2 using duration and convexity to approximate the change in price of the discount bond.

At a yield of 12.876%, the value of your investment at inception was \$4,000. If the interest rate immediately increases to 13.876%, your investment value would fall to \$3,115.20 or by \$884.80. The predicted value change using duration and convexity is

$$dB \approx B\left[-T(r-r_0) + \frac{1}{2}T^2(r-r_0)^2\right]$$
$$= 4{,}000\left[-25(0.01) + \frac{625}{2}(0.0001)\right] = -875.00$$

Note that the degree of approximation error has fallen from \$115.20 or 13.0% to −\$9.80 or −1.1%.

Discount Bonds Traded in the Marketplace

The focus now turns to discount bonds traded in the marketplace. Since we need a proxy for the zero-coupon risk-free rate of interest in subsequent chapters, we focus here on only U.S. Treasury securities. For terms to maturity of one year or less, we use Treasury bills. For terms to maturity greater than one year, we use Treasury strip bonds.

Treasury Bills A number of different zero-coupon or discount bonds trade in the U.S. Perhaps the most commonly known are U.S. Treasury bills or, simply, T-bills. To finance the operations of the government, the U.S. Treasury auctions new 28-day, 91-day, and 182-day bills every Thursday. The prices of T-bills follow certain reporting conventions. It is important to understand these reporting conventions since the interest rate on T-bills is an excellent proxy for the risk-free rate of interest—a rate applied throughout the applications of this book. Table 2.1 contains a panel of T-bill price quotes obtained from *Bloomberg* on March 29, 2006. The first column contains the maturity date of each T-bill, and the second contains the number of days to maturity. The number of days to maturity equals the actual number of days from the close on March 29, 2006 to the maturity date less one business day since T-bills have one-business day delayed settlement. The columns headed "Bid" and "Ask" are *bank discounts* or simply *discounts*. They are *neither* prices *nor* interest rates. A bank discount is defined as

$$\text{Bank discount} = (360/n)(100 - \text{T-bill price}) \tag{2.16}$$

where n is the number of days to maturity and 360 is the number of days in a "banker's year." To deduce the actual bid and ask prices for the T-bill, we must invert (2.16) and use

$$\text{T-bill price} = 100 - \text{Bank discount}(n/360) \tag{2.17}$$

If we again consider the T-bill with maturity date of 6/29/06, we see that the bid and ask discounts are 4.52 and 4.51, respectively. This means that if we bought this T-bill, you would pay

$$\text{T-bill price} = 100 - 4.51(91/360) = 98.6000\% \text{ of par}$$

If the T-bill has a par value of $1 million, you would pay $986,000.

At this juncture, it is important to digress and link the price to the continuously compounded rate of return on this T-bill. If you pay 98.83275% of par for the T-bill that matures in 69 days, the T-bill rate price promises to grow at an annualized rate of

$$r = \frac{\ln(100/98.6000)}{91/365} = 4.599\%$$

Note that 365 days rather than 360 days are used in the computation. This is because time should be measured in actual years rather than banker's years.

TABLE 2.1 U.S. Treasury bill discounts drawn from Bloomberg on March 29, 2006.

Maturity	Days to Maturity	Bid	Ask	Ask Yield
4/6/06	7	4.46	4.45	4.52
4/13/06	14	4.60	4.56	4.63
4/20/06	21	4.59	4.55	4.63
4/27/06	28	4.61	4.60	4.68
5/4/06	35	4.54	4.51	4.59
5/11/06	42	4.52	4.51	4.60
5/18/06	49	4.54	4.53	4.62
5/25/06	56	4.50	4.49	4.58
6/1/06	63	4.55	4.53	4.63
6/8/06	70	4.53	4.52	4.62
6/16/06	78	4.53	4.52	4.63
6/22/06	84	4.53	4.52	4.63
6/29/06	91	4.52	4.51	4.63
7/6/06	98	4.55	4.53	4.65
7/13/06	105	4.56	4.53	4.65
7/20/06	112	4.54	4.53	4.66
7/27/06	119	4.60	4.58	4.71
8/3/06	126	4.57	4.56	4.70
8/10/06	133	4.60	4.59	4.73
8/17/06	140	4.60	4.59	4.74
8/24/06	147	4.62	4.61	4.76
8/31/06	154	4.63	4.62	4.78
9/7/06	161	4.64	4.63	4.79
9/14/06	168	4.65	4.64	4.81
9/21/06	175	4.65	4.64	4.81
9/28/06	182	4.65	4.64	4.82

The last column is the *bond equivalent yield* based on the ask price. It represents an attempt to make the yield on a T-bill comparable to the yield on other Treasury securities whose yields are based on a 365-day, as opposed to 360-day, calendar year. Note that the reported bond equivalent yield for the 6/29/06 T-bill is 4.63%. This rate is computed by solving

$$\text{T-bill price} \times \left[1 + \text{Bond equivalent yield}\left(\frac{n}{365}\right)\right] = 100 \tag{2.18}$$

Alternatively, the bond equivalent yield may be computed directly from the T-bill's discount:

$$\text{Bond equivalent yield} = \frac{365 \times \text{Bank discount}}{360 - \text{Bank discount} \times n} \tag{2.19}$$

Either way, the number is, at best, an *approximation* for the rate of return on the T-bill. The actual rate of return (growth) of the T-bill over its life is the continuously compounded interest rate, 4.599%.

Stripped Treasury Bonds and Notes U.S. Treasury *strips*[5] are also discount bonds. The U.S. Treasury does not issue these instruments directly. Instead, they issue only coupon-bearing bonds and notes with maturities as long as 30 years. What happens is that the original issue coupon bonds are "stripped," with each coupon as well as the principal amount sold as a separate unit. In the absence of costless arbitrage opportunities, the sum of the prices of the discount bonds stripped from the original coupon issue must be equal to the price of the coupon bond.

Table 2.2 contains the ask price quotes for STRIPS of different maturities. The price data were drawn from *Bloomberg* on March 29, 2006. The last col-

TABLE 2.2 Selected U.S. Treasury STRIP prices drawn from Bloomberg on March 29, 2006.

Maturity	Ask Price	Years to Maturity	Continuous Yield
6/15/06	99.04	0.21	4.51%
9/30/06	97.68	0.51	4.63%
3/15/07	95.55	0.96	4.73%
3/15/08	91.05	1.96	4.77%
3/15/09	86.72	2.96	4.81%
3/15/10	82.91	3.96	4.73%
2/15/11	79.48	4.89	4.70%
2/15/12	75.62	5.89	4.75%
2/15/13	71.85	6.89	4.80%
2/15/14	68.30	7.89	4.83%
2/15/15	65.11	8.89	4.83%
2/15/16	61.88	9.89	4.85%
2/15/17	58.66	10.89	4.90%
2/15/18	55.54	11.89	4.94%
2/15/19	52.71	12.89	4.97%
2/15/20	49.91	13.89	5.00%
2/15/21	47.45	14.90	5.00%
2/15/22	45.23	15.90	4.99%
2/15/23	42.98	16.90	5.00%
2/15/24	40.89	17.90	5.00%
2/15/25	38.87	18.90	5.00%
2/15/26	37.09	19.90	4.98%
2/15/27	35.43	20.90	4.96%
2/15/28	33.86	21.90	4.95%

[5] The U.S. Treasury created a program called *Separate Trading of Registered Interest and Principal of Securities* (STRIPS) in February 1985 to promote liquidity in the zero-coupon bond market. For more information regarding STRIPS, see Fabozzi and Fleming (2005).

umn in the table contains the continuously compounded yield to maturity computed using equation (2.5) based on the reported ask price. The column shows that the *zero-coupon yield curve* (i.e., the relation between yield and term to maturity) is upward sloping for maturities up to about 12 years and then flattens at a level of about 5%.

COUPON-BEARING BONDS

This section focuses on coupon-bearing bonds. A *coupon-bearing bond* or, simply, a *coupon bond* pays a stated rate of interest periodically throughout the bond's life, ending with an interest payment and repayment of the bond's par value. While the valuation and risk measurement of a coupon-bearing bond is seemingly more complicated than a discount bond, it is important and useful to recognize that a coupon bond is nothing more than a portfolio of discount bonds.[6]

Valuation

The value of a coupon bond, B_c, is the sum of the values of its constituent discount bonds, that is,

$$B_c = \sum_{i=1}^{n} B_{d,i} = \sum_{i=1}^{n} CF_i e^{-r_i T_i} \tag{2.20}$$

where the subscript i denotes the ith discount bond and the value of ith discount bond is now denoted, $B_{d,i}$. CF_i is the amount of the cash flow received at the maturity of the ith discount bond, r_i is the zero-coupon discount rate used to bring the cash flow to the present, and T_i is the time until the cash flow i occurs. Prior to maturity, the cash flow equals the coupon interest payment, CF_i = COUP, as is shown in Figure 2.3. The amount of the interest payment, COUP, is the stated coupon interest rate times the par value of the bond, F_n. At maturity, the cash flow equals the coupon interest payment plus the repayment of the face value, CF_i = COUP + F_n. The number of coupon payments is denoted n. Note that equation (2.20) uses maturity-specific discount rates for each cash flow. The relation between zero-coupon yields and their terms to maturity is called the *term structure of interest rates* or the *zero-coupon yield curve*. We discuss the yield curve shortly.

FIGURE 2.3 Cash flows of a coupon-bearing bond.

[6] This valuation principle is called *valuation by replication* and is key to understanding derivative contract valuation and risk management.

ILLUSTRATION 2.4 Compute value of coupon-bearing bond given zero-coupon yield curve.

Assume that the current zero-coupon yield curve is given by the function,

$$r_i = 0.04 + 0.01\ln(1 + T_i)$$

where T_i is measured in years. Compute the value of a five-year semiannual coupon-bearing bond with a 7% coupon interest rate.

To value the bond, you need the zero-coupon interest rates corresponding to each cash flow. To do so, you apply the given term structure formula. The zero-coupon yield rate corresponding to the first constituent discount bond maturity in 0.5 years, for example, is $r_i = 0.04 + 0.01 \ln(1 + 0.5) = 4.405\%$. The cash flow promised in 0.5 years is $0.07/2 \times 100 = 3.50$, so the value of the first discount bond is $3.50e^{-0.04405(0.5)} = 3.4237$. Applying this procedure recursively (i.e., coupon bond valuation formula (2.20)), the value of the five-year, 7% coupon bond is 105.0902, the individual discount bonds of which are summarized in the table below.

Years to Maturity	Zero-Coupon Yield	Cash Flow	PV of Cash Flow
0.5	4.405%	3.50	3.4237
1.0	4.693%	3.50	3.3395
1.5	4.916%	3.50	3.2512
2.0	5.099%	3.50	3.1607
2.5	5.253%	3.50	3.0693
3.0	5.386%	3.50	2.9778
3.5	5.504%	3.50	2.8867
4.0	5.609%	3.50	2.7965
4.5	5.705%	3.50	2.7076
5.0	5.792%	103.50	77.4772
Total value			105.0902

This value may be confirmed using the OPTVAL function,

OV_IR_FIXED_ZC(*coup*, *freq*, *face*, *tb*, *ncoupr*, *term*, *rate*, *vdc*)

where *coup* is the coupon interest rate expressed in decimal form (i.e., 0.07), *freq* is the frequency of coupons per year (i.e., two), *face* is the face value of the bond (i.e., 100), *tb* is the time until the first coupon payment expressed in years (i.e., 0.5), *ncoupr* is the number of coupons remaining (i.e., 10), *term* is the vector of times to maturity of the discount bonds (i.e., the numbers in the first column in the above table), and *rate* is the vector containing the corresponding zero-coupon rates (i.e., the numbers in the second column in the above table). The indicator variable *vdc* instructs the function to return the bond's value ("*v*" or "*V*"), duration ("*d*" or "*D*"), or convexity ("*c*" or "*C*"). The bond's value, for example, is

B8 =OV_IR_FIXED_ZC($B$1,$B$2,$B$3,$B$4,$B$5,$D$3:$D$12,$E$3:$E$12,"V")

	A	B	C	D	E	F	G	H
1	Coupon rate:	7.00%		Zero-coupon yield curve				
2	Frequency:	2		Years to maturity	Yield			
3	Par value:	100		0.5	4.405%			
4	Years to first coupon:	0.5		1.0	4.693%			
5	No. of coupons remaining:	10		1.5	4.916%			
6				2.0	5.099%			
7	Fixed-rate bond			2.5	5.253%			
8	Value	105.0902		3.0	5.386%			
9	Duration	4.3174		3.5	5.504%			
10	Convexity	20.3825		4.0	5.609%			
11				4.5	5.705%			
12				5.0	5.792%			

Risk Measurement

Like in the case of discount bonds, the two most commonly used interest rate risk measures for coupon bonds are duration and convexity. In both cases, they are weighted averages of the durations and convexities of the constituent discount bonds where the weights are the proportion of coupon bond value attributable to the *i*th discount bond. Letting w_i represent the weight attributable to the *i*th discount bond, we have

$$\sum_{i=1}^{n} w_i = \frac{\sum_{i=1}^{n} B_{d,i}}{B_c} = 1 \tag{2.21}$$

Duration The duration of a coupon bond is

$$\text{DUR}_c = -\sum_{i=1}^{n} w_i \text{DUR}_{d,i} = -\sum_{i=1}^{n} w_i T_i \tag{2.22}$$

where the duration of the discount bond is given by (2.11), that is, $\text{DUR}_{d,i} = T_i$. Expression (2.22) shows that the duration of a coupon bond is a *weighted average term to maturity* of a coupon bond. Equation (2.22) also offers some important insights regarding the price risk or interest rate risk of a coupon bond. First, the longer the term to maturity of a bond, the greater the proportion of coupon bond value attributable to distant cash flows, the greater the duration, and, hence, the greater the interest rate risk. Second, the higher the coupon interest rate of a bond, the greater the proportion of the bond's value received earlier in the bond's life, the lower the duration, and, hence, the lower the interest rate risk. Third, the higher the level of interest rates, the lower importance of distant cash flows in the determination of bond value, the shorter the duration, and the lower the interest rate risk.

Convexity The convexity of a coupon bond is

$$\text{CVX}_c = \sum_{i=1}^{n} w_i \text{CVX}_{d,i} = \sum_{i=1}^{n} w_i T_i^2 \tag{2.23}$$

Like in the case of duration, the convexity of a coupon bond (2.23) is a weighted average of the convexities of the constituent discount bonds where the weights are the proportion of coupon bond value attributable to the *i*th discount bond, and the convexity of a discount bond is given by (2.15), that is, $\text{CVX}_{d,i} = T_i^2$. It is important to recognize that the duration and convexity measures (2.22) and (2.23) make the implicit assumption that the zero-coupon yield curve shifts in a parallel manner (e.g., all yields shift upward or downward by the same amount).[7]

ILLUSTRATION 2.5 Compute duration and duration/convexity approximations for a coupon bond.

Compute the actual percent change in the value of a five-year semiannual coupon-bearing bond with a 7% coupon interest rate assuming the zero-coupon yield curve changes from

$$r_i = 0.04 + 0.01\ln(1 + T_i)$$

to

$$r_i = 0.05 + 0.01\ln(1 + T_i)$$

Compare the actual percent value change with the value changes based on the duration and duration/convexity approximations.

The first step is to compute the duration and the convexity of this coupon-bearing bond. The table below details the calculations. The present value of the cash flow represented in the first row constitutes 3.258% of the total value of the coupon bond, that is,

$$\frac{3.50e^{-0.04405(0.5)}}{105.0902} = \frac{3.4237}{105.0902} = 0.03258$$

The duration of this discount bond is 0.5, so its contribution to the duration of the coupon bond is 0.03258(0.5) = 0.01629. The convexity of this discount bond is $0.5^2 = 0.25$, so its contribution to the convexity of the coupon bond is 0.03258(0.25) = 0.00814. Repeating the computations for each row, and then summing shows that the duration of the coupon bond is 4.3714 and the convexity is 20.3825.

[7] It is, of course, possible to allow the yield curve to shift in other ways. Chapter 18 focuses on the valuation of fixed income securities under different assumptions regarding the movement of interest rates through time.

Years to Maturity	Zero-Coupon Yield	Cash Flow	PV of Cash Flow	Proportion of Total	Components of Duration	Components of Convexity
0.5	4.405%	3.50	3.4237	0.03258	0.01629	0.00814
1.0	4.693%	3.50	3.3395	0.03178	0.03178	0.03178
1.5	4.916%	3.50	3.2512	0.03094	0.04641	0.06961
2.0	5.099%	3.50	3.1607	0.03008	0.06015	0.12030
2.5	5.253%	3.50	3.0693	0.02921	0.07302	0.18254
3.0	5.386%	3.50	2.9778	0.02834	0.08501	0.25502
3.5	5.504%	3.50	2.8867	0.02747	0.09614	0.33649
4.0	5.609%	3.50	2.7965	0.02661	0.10644	0.42577
4.5	5.705%	3.50	2.7076	0.02576	0.11594	0.52172
5.0	5.792%	103.50	77.4772	0.73724	3.68622	18.43111
Total			105.0902	1.0000	4.3174	20.3825

These values may be confirmed using the OPTVAL function,

OV_IR_FIXED,ZC(*coup*, *freq*, *face*, *tb*, *ncoupr*, *term*, *rate*)

whose parameters are defined above. The duration function is invoked in the spreadsheet below.

B9 =OV_IR_FIXED_ZC($B$1,$B$2,$B$3,$B$4,$B$5,$D$3:$D$12,$E$3:$E$12,"D")

	A	B	C	D	E	F	G	H
1	Coupon rate:	7.00%		Zero-coupon yield curve				
2	Frequency:	2		Years to maturity	Yield			
3	Par value:	100		0.5	4.405%			
4	Years to first coupon:	0.5		1.0	4.693%			
5	No. of coupons remaining:	10		1.5	4.916%			
6				2.0	5.099%			
7	Fixed-rate bond			2.5	5.253%			
8	Value	105.0902		3.0	5.386%			
9	Duration	4.3174		3.5	5.504%			
10	Convexity	20.3825		4.0	5.609%			
11				4.5	5.705%			
12				5.0	5.792%			

Next, compute the anticipated percentage changes in bond value based on the duration and duration/convexity approximations. Based solely on duration, the anticipated change is

$$-4.3174 \times 0.01 = -4.3174\%$$

while, based on duration and convexity, the anticipated change is

$$-4.3174(0.01) + \frac{1}{2}(20.3825)(0.0001) = -4.2155\%$$

If you simply shift the zero-coupon yield curve up by 100 basis points, you will find that the bond's value has changed from 105.0902 to 100.6585—an actual percent change of –4.2171%. Thus, you have measured the degree of approximation error for each method. The approximation based solely on duration overstates the percent movement by 0.1003%, and the approximation based on duration/convexity understates the percent movement by 0.0016%.

Coupon Bond Conventions

As noted earlier, the duration and convexity measures (2.22) and (2.23) make the implicit assumption that the zero-coupon yield curve shifts in a parallel manner (e.g., all yields shift by the same amount). To simplify matters, it is not uncommon in practice to see a single discount rate called the *yield to maturity* used to discount all cash flows of a coupon bond.

Yield to Maturity Yield to maturity is a summary statistic that describes the bond's promised rate of return. The yield to maturity is computed by setting the current bond price equal to the present value of the cash flows and solving for y, that is,

$$B_c = \sum_{i=1}^{n} CF_i e^{-yT_i} \tag{2.24}$$

Under the assumption that there is a single discount rate, the duration of a coupon bond is given by

$$\text{DUR}'_c = -\sum_{i=1}^{n} \left(\frac{CF_i e^{-yT_i}}{B_c} \right) T_i \tag{2.25}$$

and its convexity is given by

$$\text{CVX}'_c = \sum_{i=1}^{n} \left(\frac{C_i e^{-yT_i}}{B_c} \right) T_i^2 \tag{2.26}$$

Note that the duration and convexity computed using (2.25) and (2.26) are only *approximations* of the correct values (2.22) and (2.23). The present value of the ith cash flow is not equal to the price of the ith discount bond, that is,

$$CF_i e^{-yT_i} \neq B_{d,i}$$

The OPTVAL Function library contains a function for computing the value, the duration, and the convexity of a fixed rate bond given its yield to maturity:

OV_IR_FIXED_YLD(*coup*, *freq*, *face*, *tb*, *ncoupr*, *yld*, *vdc*)

where *coup* is the coupon interest rate expressed in decimal form, *freq* is the frequency of coupons per year, *face* is the face value of the bond, *tb* is the time until the first coupon payment expressed in years, *ncoupr* is the number of coupons remaining, and *yld* is the bond's promised yield to maturity. The indicator variable *vdc* instructs the function to return the bond's value ("*v*" or "*V*"), duration ("*d*" or "*D*"), or convexity ("*c*" or "*C*").

ILLUSTRATION 2.6 Compute yield to maturity of coupon-bearing bond given the yield curve.

Assume that the current zero-coupon term structure of spot rates is given by the curve,

$$r_i = 0.04 + 0.01\ln(1 + T_i)$$

where T_i is measured is years. Compute the value and the yield to maturity of a five-year semiannual coupon-bearing bond with a 7% coupon interest rate. If this coupon-bearing bond can be purchased for \$104, can you earn a costless arbitrage profit, and, if so, how?

You know from Illustration 2.4 that the five-year, 7% bond is 105.0902. The yield to maturity of this bond is computed by setting the bond price equal to the present value of the cash flows and solving for a single discount rate. The discount rate that satisfies

$$105.0902 = \sum_{i=1}^{9} 3.50e^{-yT_i} + 103.50e^{-yT_{10}}$$

is 5.729% as is shown in the table below. The syntax for the OPTVAL function is

OV_IR_FIXED_YLD_YIELD(*coup*, *freq*, *face*, *tb*, *ncoupr*, *bprce*)

where all parameters are defined as above and *bprce* is the bond's price including accrued interest.

B7 =OV_IR_FIXED_YLD_YIELD($B$1,$B$2,$B$3,$B$4,$B$5,$B$6)

	A	B	C	D	E	F
1	Coupon rate:	7.00%				
2	Frequency:	2				
3	Par value:	100				
4	Years to next payment:	0.5				
5	No. of payments remaining:	10				
6	Bond price:	105.0902				
7	Implied yield to maturity:	5.729%				

Note that this yield to maturity of the coupon bond is below the zero-coupon rate on a five-year zero-coupon bond, 5.792%, in Illustration 2.5. This is because a five-year coupon-bearing bond does not have five years to maturity from an economic standpoint. The intermediate payments made during the bond's life effectively shorten its overall maturity.

Assuming the coupon-bearing bond can be purchased for \$104, a costless arbitrage profit can be earned. To do so, you would buy the coupon bond and then sell zero-coupon bonds in the amount and maturity of each cash flow, that is, sell 3.50 in par value of zero-coupon bonds maturing in six months, and 3.50 in par value of zero-coupon bonds maturing in one year, and so on. In this way, the interest receipts of the coupon-bearing bond exactly match the payments you need to make to cover your short sale obligations. Since you know that you can buy the coupon bond for \$104 and sell the zero-coupon bond portfolio (using the zero-coupon yield curve) for \$105.0902, the present value of the costless arbitrage profit of \$1.0902.

ILLUSTRATION 2.7 Compute duration and convexity of coupon bond using yield to maturity.

Again, assume that the current zero-coupon yield curve is given by the function,

$$r_i = 0.04 + 0.01\ln(1 + T_i)$$

where T_i is measured is years. Compute the duration and convexity of a five-year semiannual coupon-bearing bond with a 7% coupon interest rate using the single yield to maturity, 5.729%, from Illustration 2.6.

The table below summarizes the computations from basic principles. First, you compute the present values of the cash flows using a constant yield to maturity. Naturally, the total of the values of the discount bonds computed using yield to maturity is 105.0902. Recall from Illustration 2.6, this is exactly how the yield to maturity was defined. Next, you compute the proportion of total coupon bond value that is attributable to each discount bond. The first row of the table shows

$$3.4012e^{-0.05729(0.5)}/105.0902 = 0.01618$$

Finally, compute the contributions of each discount bond to the duration and convexity of the coupon bond and sum as you did in Illustration 2.4. The yield-based duration is 4.3240, compared with 4.3174 using the zero-coupon yield curve approach, and the yield-based convexity is 20.4273, compared 20.3825 using the zero-coupon yield curve approach. While these differences are small in the illustration at hand, they will vary depending on factors such as the coupon rate of the bond, its term to maturity, and the slope of the yield curve.

Years to Maturity	Cash Flow	PV of Cash Flow	Proportion of Total	Components of Duration	Components of Convexity
0.5	3.50	3.4012	0.03236	0.01618	0.00809
1.0	3.50	3.3051	0.03145	0.03145	0.03145
1.5	3.50	3.2118	0.03056	0.04584	0.06876
2.0	3.50	3.1211	0.02970	0.05940	0.11880
2.5	3.50	3.0329	0.02886	0.07215	0.18038
3.0	3.50	2.9473	0.02805	0.08414	0.25241
3.5	3.50	2.8640	0.02725	0.09539	0.33385
4.0	3.50	2.7832	0.02648	0.10593	0.42374
4.5	3.50	2.7046	0.02574	0.11581	0.52115
5.0	103.50	77.7191	0.73955	3.69773	18.48867
Total		105.0902	1.0000	4.3240	20.4273

Risk Management

Risk management is the general theme of this book. Although the purpose of this chapter is to lay the foundation for risk management using derivatives, it is instructive to introduce the concept of hedging at this juncture to reinforce the use of the bond risk management tools of duration and convexity.

Risk has a number of definitions. For now, assume that risk refers to unanticipated changes in the value of an asset that we hold. *Hedging* refers to reducing the risk of our position by buying or selling other assets whose collective value changes by the same amount as the value of the asset we hold. In the context of bonds and interest rate risk measurement, a perfect hedge is one whose value changes in an equal and opposite direction, that is,

$$\frac{d\text{Value of unhedged position}}{dr} = \frac{d\text{Value of hedge instruments}}{dr} \tag{2.27}$$

Duration and convexity provide the means for measuring the value changes of your portfolio and the hedge instruments should interest rates change. To completely hedge interest rate risk exposure means finding the number of units of the hedge instrument to buy or sell such that the value of the overall hedged portfolio does not change if interest rates change, that is,

$$dB_P + n_H dB_H = 0 \tag{2.28}$$

where B_P is the value of your bond position and B_H is the value of one unit of the hedge instrument, where the expression dr has been dropped because it is common to both sides of the equation. Duration-based hedging means approximating the changes of value with the product of duration and bond value. The number of units of the hedge instrument to buy or sell is therefore determined by solving

$$\text{DUR}_P B_P + n_H \text{DUR}_H B_H = 0 \tag{2.29}$$

where $\text{DUR}_P(\text{DUR}_H)$ is the duration of the unhedged bond portfolio (hedge instrument) and $B_P(B_H)$ is the market value of the unhedged bond portfolio (market value of the hedge instrument). Rearranging (2.29) to solve for the number of hedge bonds n_H, we get

$$n_H = -\frac{\text{DUR}_P B_P}{\text{DUR}_H B_H} \tag{2.30}$$

ILLUSTRATION 2.8 Hedge interest rate risk of bond portfolio using duration.

Suppose you own \$30 million in par value of a 10% coupon-bearing bond with 10 years to maturity. Its current yield to maturity is 8%. Suppose also that you expect that interest rates may increase over the next few days and want to hedge your interest rate risk exposure. Unfortunately, the bond you hold does not have a liquid market and selling quickly is impossible. You have the opportunity to sell a more liquid bond, however. Its coupon rate is 9%, term to maturity is 12 years, par value is \$100,000, and yield to maturity is 7%. How many bonds should you sell? Assume both bonds pay coupons semiannually with the first coupon being paid in exactly six months. Show how effective the hedge is by plotting the changes in the hedged portfolio value over a range of yield changes from –5% to +10%.

The first step is to compute the value and the duration of the bonds. Since you have no information about the zero-coupon yield curve, you can use the yield-based computations (2.24) and (2.25). And, rather than go through the algebra, use the OPTVAL functions. The value and durations of the unhedged bond position and the hedge instrument are as follows:

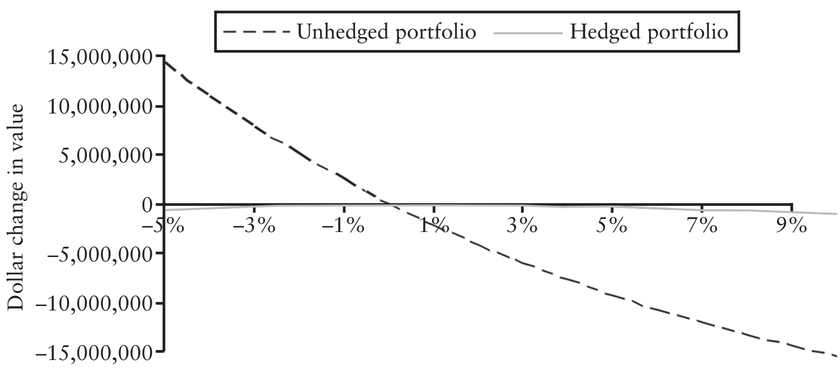
C11 =OV_IR_FIXED_YLD($C$3,$C$4,$C$5,$C$6,$C$7,$C$8,"d")

	A	B	C	D	E
1		Bond	Hedge		
2		portfolio	instrument		
3	Coupon rate	10.00%	9.00%		
4	Frequency	2	2		
5	Par value	30,000,000	100,000		
6	Years to next payment	0.5	0.5		
7	Number of payments	20	24		
8	Yield to maturity	8.00%	7.00%		
9					
10	Value	33,719,782.77	114,965.65		
11	Duration	6.7539	7.8916		

The number of the hedge bonds to sell to immunize your portfolio from interest rate movements is therefore

$$n_H = -\frac{6.7539(33{,}719{,}782.77)}{7.8916(114{,}965.65)} = -251.019$$

To test the effectiveness of the hedge, compute (a) the change in value of the unhedged bond portfolio, and (b) the change in value of the hedged portfolio using a range of yield changes from –5% to +10%. These changes in value are shown in the figure below. As the figure shows, a yield increase produces a significant decline in the unhedged portfolio value. A yield increase of 200 basis points reduces bond portfolio value by more than $4,000,000. After the hedge is in place, however, a yield increase causes the hedged portfolio value to fall by about $63,000 (which cannot be detected on the figure because of the scale). The fact that the hedged portfolio value changes are not 0 across *all* levels of yield change means that the hedge is not fully effective. Recall that the duration-based hedge fails to account for the convexity of the bond valuation formula. Accounting for both duration and convexity will improve the hedging effectiveness.



Hedging effectiveness can be improved by incorporating both duration and convexity components of bond value change. In order to do so, however, two hedge instruments will be required. To identify the appropriate number of hedge bonds to buy or sell, you will need to match the duration and the convexity of the bond portfolio that you want to hedge with the duration and convexity of the hedge instruments. To negate the duration risk of the portfolio, you must satisfy the duration constraint

$$\text{DUR}_P B_P + n_{H,1}\text{DUR}_{H,1}B_{H,1} + n_{H,2}\text{DUR}_{H,2}B_{H,2} = 0 \tag{2.31}$$

Equation (2.31) is the counterpart to (2.29) in which only duration risk was considered. The constraint merely says that you do not have any duration risk exposure after setting $n_{H,1}$ and $n_{H,2}$. Simultaneously, you must also satisfy the convexity constraint,

$$\text{CVX}_P B_P + n_{H,1}\text{CVX}_{H,1}B_{H,1} + n_{H,2}\text{CVX}_{H,2}B_{H,2} = 0 \tag{2.32}$$

where CVX refers to convexity of the different instruments and the subscripts 1 and 2 refer to the first and second hedge instruments. Since there are two equations (i.e., (2.31) and (2.32)) and two unknowns ($n_{H,1}$ and $n_{H,2}$), we can solve uniquely. The solution can be found algebraically or computationally using a iterative technique such as Microsoft Excel's SOLVER.

ILLUSTRATION 2.9 Hedge interest rate risk of bond portfolio using duration/convexity.

Use the same problem information as in Illustration 2.8. In addition, assume that a second hedge bond is available. Its coupon rate is 5%, term to maturity is 20 years, par value is $100,000, and yield to maturity is 7.5%. How many of each hedge bonds should you sell if you want to hedge both the duration and convexity risk of your portfolio? Show how effective the duration/convexity hedge is relative to the duration-only hedge by plotting the changes in the hedged portfolio value over a range of yield changes from –5% to +10%.

The first step is to compute the value, and convexity of the bonds. The information is summarized below.

D12 =OV_IR_FIXED_YLD(D$3,D$4,D$5,D$6,D$7,D$8,"c")

	A	B	C	D
1		Bond	Hedge	Hedge
2		portfolio	instrument 1	instrument 2
3	Coupon rate	10.00%	9.00%	5.00%
4	Frequency	2	2	2
5	Par value	30,000,000	100,000	100,000
6	Years to next payme	0.5	0.5	0.5
7	Number of payments	20	24	40
8	Yield to maturity	8.00%	7.00%	7.50%
9				
10	Value	33,719,782.77	114,965.65	73,139.32
11	Duration	6.7539	7.8916	11.5501
12	Convexity	57.4665	79.6506	185.5095

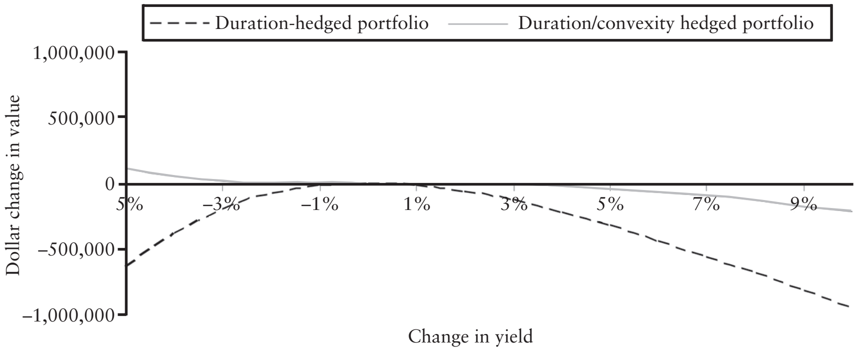
The system of equations (2.31) and (2.32) are:

$$6.7539(33{,}719{,}782.77) + n_{H,1}(7.8916)(114{,}965.65) + n_{H,2}(11.5501)(73{,}139.32) = 0$$

and

$$57.4665(33{,}719{,}782.77) + n_{H,1}(79.6506)(114{,}965.65) + n_{H,2}(185.5095)(73{,}139.32) = 0$$

The solution for the risk-free hedge is $n_{H,1} = -317.661$ and $n_{H,2} = 71.572$. The effectiveness of the duration/convexity hedge vis-à-vis the duration-only hedge is shown in the figure below. For small changes in yield, the hedges perform about the same. For large changes in yield, however, the duration/convexity hedge clearly outperforms.



Coupon Bonds Traded in the Marketplace

Probably the most widely known coupon bonds are the *T-bonds* and *T-notes* issued by the U.S. Treasury. Both are coupon bonds—the difference between them is that T-notes are originally issued with two to 10 years to maturity and T-bonds are originally issued with maturities longer than 10 years. On August 2001, the U.S. Treasury suspended the periodic auctioning of the 30-year bond. In August 2005, the Treasury announced its reintroduction. The first auction after the reintroduction was held on February 9, 2006. This issue is the 4½ Feb 2036 that appear in Table 2.3.

Table 2.3 contains U.S. Treasury bond and note prices on March 29, 2006. A number of reporting conventions appear. First, coupon bond prices are reported with a dash rather than a decimal. This is because the digits to the right of the dash represent the number of 32nds rather than the number of 100ths. A price of 99-16 implies 99.50% of par. Where the price has the suffix "+," an additional one-half 32nds is added to the price. A price of 99-16+ is, therefore, $99\frac{33}{64}$ or 99.515625% of par.

A second convention, although not stated in the panel of prices reported in the table, is that coupon payments are semiannual (i.e., occur each 6 months). The "6¼s of May 2030," for example, pay coupon interest of 3.125% of par on November 15 and May 15 each year through the bond's life. The last coupon and the face value are paid on May 15, 2030.

A third convention is that the *reported* or *quoted* price of the T-bond or T-note excludes *accrued interest* during the current coupon period. Accrued interest equals the amount of the semiannual coupon payment times the proportion of the current coupon period that has elapsed since the last coupon payment, that is,

$$AI = \text{COUP}\left(\frac{\text{Number of days since last coupon was paid}}{\text{Total number of days in current coupon period}}\right) \tag{2.33}$$

TABLE 2.3 Selected U.S. Treasury bond and note prices drawn from Bloomberg on March 29, 2006.

Rate	Maturity	Bid	Ask	Ask Yield	Notes
2¾	Jun 2006	99-16+	99-17	4.58	
2⅞	Nov 2006	98-22	98-22+	4.86	
3¾	Mar 2007	98-28+	98-29+	4.86	
4⅝	Mar 2008	99-20	99-20+	4.82	2-year
4½	Feb 2009	99-05	99-05+	4.81	3-year
4½	Feb 2011	98-21+	98-22	4.80	5-year
13⅞	May 2006-11	101-03+	101	4.49	callable
4⅞	Feb 2012	100-09	100-10	4.81	
11¾	Nov 2009-14	122-20	122-28	8.02	callable
4¼	Nov 2014	95-25	95-26	4.85	
4¼	Aug 2015	95-19+	95-20+	4.83	
4½	Nov 2015	97-12+	97-14	4.84	
4½	Feb 2016	97-20	97-20+	4.80	10-year
9⅛	May 2018	137-20	137-22+	4.95	
8⅛	Aug 2021	132-27+	132-28+	5.03	
7⅝	Feb 2025	131-14	131-15+	5.02	
6⅛	Nov 2027	114-15	114-17	5.02	
6¼	May 2030	118-00	118-01+	4.96	
5⅜	Feb 2031	106-11	106-13+	4.92	
4½	Feb 2036	94-31+	95-00	4.82	30-year

The quoted bond price is reported as its current price less accrued interest. Thus, if we purchase the bond today, we pay the reported price plus accrued interest. This practice seems silly. It is! But, it was instituted many decades ago, and traditions are sometimes hard to break. In the parlance of bond traders, the price excluding accrued interest is called the "clean price," and the price including accrued interest is called the "dirty price," "gross price," or "full price."[8,9]

A fourth convention is that Treasury bonds with hyphenated maturity dates are callable. Table 2.3 has two such issues. The notation "13⅞ May 2006-14" means that the U.S. Treasury has the right to call all bonds back at any of the coupon dates between May 15, 2006 and May 15, 2011. Given the high coupon of this issue, it should not be surprising to learn that, on January 13, 2006, the U.S. Treasury called for redemption of this issue at par on May 15, 2006. Consequently, it is being priced as if its term to maturity is about two months. Compare its promised yield to, say, the 2¾ Jun 2006 issue as opposed to the 4⅞ Feb

[8] This "actual/actual" definition of accrued interest applies only to Treasury notes and bonds. Accrued interest for corporate and municipal bonds is based on a 360-day year, with each month having 30 days, and is referred to as being on a "30/360" basis.

[9] Like Treasury bills, Treasury notes and bonds have a one business day settlement convention. Corporate bonds, on the other hand, generally have three-day settlement.

2012 issue. The 11½ Nov 2009-14 issue is also callable. Since this call option has value to the Treasury, its price (yield) will be less (greater) than comparable issues with no call feature. Note that the 11½ Nov 2009-14 have a higher promised yield to maturity than comparable maturity bonds in the table.

Finally, it is worth noting that, while the market for Treasuries is extremely active, the most recent issues, called *on-the-run securities*, have the highest trading volume. This can be seen in Table 2.3. The bonds and notes denoted by "*n*-year" in the last column are on-the-run issues. Note that the spreads between the bid and ask price quotes are smaller for these issues than for the off-the-run issues. Holding other factors constant, the higher the trading volume, the lower the bid/ask spread.

ILLUSTRATION 2.10 Deduce price of coupon-stream.

In Table 2.2, the strip bond maturing in February 2016 has a reported ask price of 61.88. In Table 2.3, the 4½ Feb 2016 issue has a reported ask price of 97.20+. Deduce the price of the coupons of the 4½ Feb 2016 without using the bond valuation formula.

First, we need to compute the decimal price of the 4½ Feb 2016 coupon-bearing bond. The reported ask price in Table 2.3 is 97-20+, which translates to $97\frac{41}{64}$ or 97.6406% of par. The number of days that have elapsed in the current coupon period as of March 29, 2006 is 42, and the total number of days in the current coupon period is 184. The accrued interest is, therefore,

$$(4.50/2) \times (42/181) = 0.5221$$

and the full price of the bond is 97.6406 + 0.5221 = 98.1627. Second, by the law of one price, the present value of the principal of the coupon-bearing bond is 61.88% of par. Consequently, the price of the coupon stream is 36.2827. To summarize,

	Price	
	In 32nds	In Decimal
Coupon-bearing bond		
Quoted bond price:	97.205	97.6406
Accrued interest:		0.5221
Market price of bond:		98.1627
Strip bond		
Quoted bond price:		61.8800
PV of coupon payments:		36.2827

ILLUSTRATION 2.11 Compute price of call feature in coupon-bearing bond.

Suppose that you observe the following U.S. Treasury bond prices (quoted in 32nds):

Coupon Rate	Maturity	Price
8¼%	May 15, 2010-15	103-19
12%	May 15, 2015	133-13
0%	May 15, 2015	47-14

The 8¼% bond is callable at par on any May 15 in the years 2010 through 2015. Based on the reported prices, compute the value of the embedded call feature.

The value of the call feature can be deduced by using the valuation by replication principle. From the problem information, you can create an 8¼% coupon-bearing, noncallable bond from the 12% coupon-bearing bond and the zero-coupon bond. To reproduce the 8¼% coupon payments, you need to buy

$$\frac{8.25}{12} = 0.6875 \text{ units}$$

of the 12% bond. While this purchase creates the desired coupon stream, the repayment of the principal in May 2015 will amount to only 68.75. To make up for the difference, 100 – 68.75 = 31.25, you need to buy 0.3125 units of the zero-coupon bond. Thus, in the absence of costless arbitrage opportunities, the price of an 8¼% coupon-bearing noncallable bond is

$$0.6875 \times 133.40625 + 0.3125 \times 47.43750 = 106.5410$$

The value of the call feature is, therefore, 106.5410 – 103.59375 = 2.9473.

Bond Equivalent Yield The continuously compounded yield to maturity of the 6¼s of May 2030 can be computed using equation (2.24) and is 5.5847%.[10] In Table 2.2, however, the yield to maturity of the 6¼s of May 2030 is reported as 5.66%. The reported rate is called a *bond equivalent yield*. While bond equivalents yield are not used is any of the subsequent chapters, it is useful to know the conventions that bond markets have adopted, if only to be able to reconcile market reporting with actual economic values.

The bond equivalent yield, y_s, is a nominal yield. It is determined by equating the market price of a bond to the present value of its cash flows and solving for y_s, that is,

$$B_c = (1 + y_s/2)^{-n_{dr}/n_{dcp}} \sum_{i=0}^{n-1} CF_i (1 + y_s/2)^{T_i} \tag{2.34}$$

where n_{dr} is the number of days remaining in the current coupon period, and n_{dcp} is the total number of days in the current coupon period.[11] In essence, what the right-hand side of (2.34) does is have you go forward until the date of the next coupon payment and value the bond, and then discount the value at the time of the next coupon using a discount factor that depends on the fraction of the current coupon period remaining. Note that when you compute the value of the bond at the time of the next coupon, the first coupon in the summation does not get discounted since it is being paid immediately. It is also worth noting that the reported bond equivalent yield is based on the ask price (rather than the bid

[10] The continuously compounded yield to maturity may be computed using OV_IR_FIXED_YIELD(*cdat*, *lcpn*, *ncpn*, *coup*, *mdat*, *bprce*), where *cdat* is the current date, *lcpn* is the last coupon date, *ncpn* is the next coupon date, *coup* is the coupon rate expressed in decimal, *mdat* is the maturity date of the bond, and *bprce* is the bond price including accrued interest.

[11] Again, this convention applies to Treasury bonds and notes only. Corporate and municipal bonds have a different day count convention.

price). The rationale is that, if you bought the bond, the bond equivalent yield is the approximate rate of return you would earn if you held it to maturity. The bond equivalent yield of the 6¼s of May 2030, for example, is determined by

$$110.2092 = (1 + y_s/2)^{-76/184} \sum_{i=0}^{60-1} CF_i(1 + y_s/2)^{T_i}$$

where CF_i equals the coupon interest payment, 3.125, in each period but the last and is the coupon interest payment plus the repayment of principal at maturity, 103.125.

TERM STRUCTURE OF INTEREST RATES

The rates reported for the discount bonds in Tables 2.1 and 2.2 reveal that the zero-coupon interest rate (or *spot* rate of interest[12]) varies with term to maturity. The relation between spot rates and term to maturity is called (interchangeably) the *term structure of interest rates*, the *term structure of spot rates*, and the *zero-coupon yield curve*. Depending on the economic environment, the nature of the relation may change.[13] Note that it is important that all bonds used in examining the term structure of interest rates must have a common degree of default risk. We do not want the relation between yield and term to maturity to be obfuscated by the fact that yields also vary with risk. In practice, the shape of the zero-coupon yield curve is determined using the rates from U.S. Treasury instruments like those reported in Tables 2.1 and 2.2. Treasury securities are all viewed as being free from default risk.

In applying the coupon bond valuation formula (2.20), it is necessary to know the zero-coupon rate for each cash flow. The cash flows of a coupon bond, however, may fall between the maturities of the zero-coupon rates observed in the marketplace. Suppose, for example, that the bond you are valuing has a cash flow occurring in four months, and you can find only zero-coupon rates with three months and six months to maturity. Somehow, you have to come up with a four-month zero-coupon rate. One method is *linear interpolation*. You would simply take a time-weighted average of the three-month and six-months rates, weighting the three-month rate with 2/3 and the six-month rate with 1/3. Another approach is to smooth the entire set of zero-coupon rates at once using techniques such as *ordinary least squares regression* or *cubic spline interpolation*. Such techniques are described in detail in Chapter 18.

For illustrative purposes, we assume that the entire term structure of observed rates has been smoothed and can be represented by a mathematical relation such as

$$r_i = 0.04 + 0.01\ln(1 + T_i) \tag{2.35}$$

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[12] It is called the *spot* rate of interest because it applies to a loan that begins today.

[13] Typically, the curve is upward sloping because lenders of funds prefer short maturities while borrowers prefer long.

where r_i is the continuously compounded rate on a loan maturing in T_i years. Where T_i is 0, the rate is 4%. This is the rate of interest on an overnight loan. Where T_i is 5, the rate of interest is 5.792%. Figure 2.1 shows the rates produced by (2.35) for different terms to maturity. As the figure shows, the term structure is upward sloping, with the rate of increase diminishing with term to maturity. Also plotted in Figure 2.1 are the discount factors corresponding to each zero-coupon rate. Many practitioners prefer working with discount factors rather than discount rates. Recall that a discount factor is today's price of $1 received at future time T_i, that is,

$$DF_i = e^{-r_i T_i}$$

Implied Forward Rates of Interest

The zero-coupon yield curve represents the spot rates interest on loans of varying maturities. The loans begin *today* and extend until the end of the bond's life, T. The zero-coupon yield curve also embeds information about the rates of interest that may be earned on loans in the *future*. Such rates are called *forward rates of interest*. To deduce the forward rate on a loan that will begin at time T_1 and run until time T_2, we first go to the zero-coupon yield curve and find the spot rates corresponding to each maturity, that is, r_1 and r_2. Next, assume that we want to invest $1 for a period of time equal to T_2. One way we can do this is to buy a zero-coupon bond with maturity T_2. Another way is to buy a zero-coupon bond with maturity T_1, and then reinvest the terminal proceeds in a zero-coupon bond with maturity $T_2 - T_1$. The forward rate of interest from T_1 to T_2 can be deduced by equating the terminal values of the two investment alternatives, that is,

$$e^{r_2 T_2} = e^{r_1 T_1} e^{f_{1,2}(T_2 - T_1)} \tag{2.36}$$

Taking the natural logarithm of both sides of (2.36), replacing subscript 1 with the notation i and 2 with j, and rearranging to isolate $f_{i,j}$, the *implied forward rate of interest* on a loan beginning at time T_i and ending at time T_j is

$$f_{i,j} = \frac{r_j T_j - r_i T_i}{T_j - T_i} \tag{2.37}$$

The zero-coupon yield curve can also be used to deduce *forward discount factors*. From (2.36), we know

$$\frac{1}{DF_2} = \frac{1}{DF_1} \times \frac{1}{FDF_{1,2}} \tag{2.38}$$

where DF_i is the discount factor of the ith zero-coupon bond currently observed in the marketplace and $FDF_{i,j}$ is the *implied forward discount factor* beginning at time T_i and ending at time T_j. Rearranging (2.38), we see that, in general, implied forward discount factors may be computed as

$$FDF_{i,j} = \frac{DF_j}{DF_i} \tag{2.39}$$

Consequently, implied forward rates may also be computed as

$$f_{i,j} = \frac{\ln(DF_i/DF_j)}{T_j - T_i} \tag{2.40}$$

ILLUSTRATION 2.12 Compute forward rates and forward discount factors from zero-coupon yield curve.

Assume that the current zero-coupon term structure of spot rates is given by the curve,

$$r_i = 0.04 + 0.01\ln(1 + T_i)$$

Compute the spot rates and discount factors on loans beginning now and ending in years 1 through 10, by increments of one year. Also, compute the one-year forward rates and one-year forward discount factors beginning at the end of years 1 through 9 by increments of one year.

To compute the zero-coupon spot rates, apply the given term structure formula. The one-year spot rate, for example, is $r_1 = 0.04 + 0.01\ \ln(1 + 1) = 4.693\%$. The one-year discount factor is $DF_1 = e^{-0.04693(1)} = 0.9542$. The complete set of results is shown in the table below. To compute the forward rates and forward discount factors based on the zero-coupon spot rates, you apply the formula (2.37) and (2.39). The implied forward rate on a one-year loan beginning in 1 year is

$$f_{1,2} = \frac{0.05099(2) - 0.04693(1)}{2 - 1} = 5.504$$

The implied price of a one-year discount bond paying \$1 in year 2 is

$$FDF_{1,2} = \frac{DF_2}{DF_1} = \frac{0.9031}{0.9542} = 0.9464$$

Note also the relation between the implied forward rate and the implied discount factor, that is,

$$f_{1,2} = \frac{\ln(0.9464)}{2 - 1} = 5.504\%$$

The OPTVAL Function library includes a routine for computing implied forward rates:

OV_IR_TS_FORWARD_RATE(*r*1, *r*2, *t*1, *t*2)

where *r*1 and *r*2 are the spot rates maturing at the beginning and at the end of the forward rate period, and *t*1 and *t*2 are the times to maturity of the respective rates. An application of the function is shown in the spreadsheet below.

D5 =OV_IR_TS_FORWARD_RATE(B5,B6,A5,A6)

	A	B	C	D	E	F
1				One-year	Forward	
2	Years to	Spot	Discount	forward	discount	
3	maturity	rate	factor	rate	factor	
4	0	4.000%	1.0000			
5	1	4.693%	0.9542	5.504%	0.9464	
6	2	5.099%	0.9031	5.962%	0.9421	
7	3	5.386%	0.8508	6.279%	0.9391	
8	4	5.609%	0.7990	6.521%	0.9369	
9	5	5.792%	0.7486	6.717%	0.9350	
10	6	5.946%	0.6999	6.881%	0.9335	
11	7	6.079%	0.6534	7.022%	0.9322	
12	8	6.197%	0.6091	7.145%	0.9310	
13	9	6.303%	0.5671	7.256%	0.9300	
14	10	6.398%	0.5274			

Note that the implied one-year forward rate starting at time 0 equals the one-year spot rate. This stands to reason since a forward rate loan beginning at time 0 is simply a spot rate loan. Note also that the implied forward rates can be significantly higher than the spot rates. The spot rates on 9-year and 10-year loans are 6.303% and 6.398%, respectively, and yet the implied one-year forward rate for a loan beginning at the end of year 9 is 7.256%.

ILLUSTRATION 2.13 Lock-in interest rate on forward loan.

Suppose that you go to your local bank and tell the manager that you want to borrow $50,000 in three months and want to repay the loan with a single balloon payment nine months later. Because you believe interest rates will rise over the next three months, you further request that the interest be locked-in today. The manager says that your credit risk is no problem, but that he cannot lock-in the interest rate because he has no idea what it will be in three months. You then ask about the current borrowing and lending rates at the bank, and he gives you the following table.

Term	Lending Rate	Borrowing Rate
3 months	3.00%	3.50%
6 months	3.50%	4.00%
9 months	4.00%	4.50%
1 year	4.50%	5.00%

Based on these quoted rates, what forward rate can you lock in today on a nine-month loan beginning in three months? Show how to structure the forward loan. What rate can you lock in today? (Assume all interest rates are continuously compounded.)

In order to compute the forward rate, you must identify the two spot rates that straddle the forward period, that is, the spot rates that mature at the beginning and end of the forward loan period—three months and one year. Because you want to borrow money in the forward period, the longer term spot rate needs to be a borrowing rate, 5.00%. Since you do not need the loan over the first three months of the year, the shorter term spot rate is the lending rate, 3.00%. Thus, the implied forward rate of interest on a nine-month loan beginning loan beginning in three months is

$$f_{0.25,\,1} = \frac{0.05(1) - 0.03(0.25)}{1 - 0.25} = 5.667\%$$

In order to structure the forward loan, you must figure out how much to borrow. Recall that you need to borrow \$50,000 in three months and want to repay the loan nine months later. To provide for the \$50,000 cash inflow, you need to lend the present value of the \$50,000 in three months. The rate that you will earn on such a deposit is 3.00%. The present value of \$50,000 received at the end of three months is

$$50{,}000e^{-0.03(0.25)} = 49{,}626.40$$

But where do you get the needed deposit of \$49,626.40? The answer is that you borrow that amount for a year. By borrowing \$49,626.40 for one year and lending that same amount for three months, you have synthetically structured a nine-month forward loan beginning in three months. The net cash flows of the agreement are certain and are as follows:

Action	Today	3 Months	1 Year
Borrow	49,626.40		–52,170.80
Lend	–49,626.40	50,000.00	

The rate on the forward loan is ln(52,170.80/50,000)/0.75 = 5.667%.

STOCK VALUATION

Shares of stock are pieces of the ownership of a corporation. Shareholders derive value in two ways, through periodic cash dividend payments and through any price appreciation (or depreciation) that may occur while holding the stock. Valuing a stock is like valuing a coupon-bearing bond in the sense that both are present values of expected future cash flow streams. Unlike a bond, however, the expected periodic cash flows (i.e., dividend payments) are not specified in a contract. Moreover, absent bankruptcy, the life of a stock is infinite.

In the stock valuation problem, the expected future cash flows are cash dividends. We denote D_i as the ith future cash dividend, where the dividend stream continues indefinitely, that is, D_1, D_2, D_3, ... The time from now until the ith dividend is received is denoted t_i. The current dividend, D_0, is assumed to have just been paid. The present value of all expected future cash dividends is

$$S = \sum_{i=1}^{\infty} D_i e^{-kt_i} \tag{2.41}$$

where k is the required rate of return on the stock.[14]

Equation (2.41) is a *stock valuation formula*. On first appearance, it may seem appropriate only for those individuals who plan to hold the stock indefinitely, but

[14] For expositional convenience, the rate of return k is assumed to be the same for each cash dividend payment. There is no reason in principle, however, that the discount rate cannot be a function of time.

that is not the case. Even if our anticipated holding period is much shorter, (2.41) remains an appropriate model. To see this, consider the value that we would assign the stock if we anticipated selling it after the *n*th dividend is paid, that is,

$$S = \sum_{i=1}^{n} D_i e^{-kt_i} + S_n e^{-kt_n} \tag{2.42}$$

where S_n is the expected share price at time t_n. To develop an expectation of the expected share price at time t_n, assume that we sell the stock to someone who plans to hold it indefinitely. The trade price will be

$$S_n = \sum_{i=n+1}^{n} D_i e^{-k(t_i - t_n)} \tag{2.43}$$

Substituting (2.43) into (2.42) and simplifying, we are back to (2.41).

Constant Dividend Growth

As a practical matter, the valuation equation (2.41) is difficult to implement since it requires that we estimate the cash dividend amounts from next period through infinity. What is more common in practice is to estimate next period's cash dividend, D_1, and then assume that subsequent dividends grow at a constant rate. Assuming dividends grow at a continuous constant rate, g, we can rewrite (2.41) as

$$S = \sum_{i=1}^{\infty} D_i e^{-(k-g)t_i} \tag{2.44}$$

As it turns out, (2.44) is the sum of an infinite geometric progression whose value is easily computed, as is demonstrated in Appendix 2B. The common ratio, b, in Appendix 2B is $b = e^{-(k-g)}$, so $1/b = e^{k-g}$ and the per share value of the common stock[15] is

$$S = \frac{D_0}{e^{k-g} - 1} \tag{2.45}$$

ILLUSTRATION 2.14 Value common stock with constant dividend growth.

Suppose you are considering buying a particular stock at its current price of \$15 a share. The stock just paid a dividend of \$2 a share. Based upon your historical dividend analysis, you expect the stock's dividend to grow at a constant continuous rate of 1% a year indefinitely,

$$D_t = 2e^{0.01t}, \text{ for } t = 1, 2, 3, \ldots$$

[15] This is one variation of what is often referred to as the Gordon (1962) constant growth model.

where the dividends are paid annually. Based on your risk analysis, you believe the required rate of return for the stock is 12%. Should you buy the stock?

Based on the parameters you have estimated, the stock's value is

$$S = \frac{D_0}{e^{k-g} - 1} = \frac{2}{e^{0.12 - 0.01} - 1} = 17.20$$

Consequently, the stock is under-priced and should be purchased.

SUMMARY

Effective risk management requires precise risk measurement, and precise risk measurement requires a thorough understanding of security valuation. This chapter provides the foundations of security valuation. The first section discussed the two key assumptions underlying valuation—the absence of costless arbitrage opportunities and frictionless markets. The first assumption is predicated on the notion that individuals prefer more wealth to less. It is essential. The second assumption is one of convenience. It allows security valuation models to be developed in an unencumbered fashion. We relax this assumption in various ways as we proceed through the remaining chapters in the book.

The next five sections focus on the time value of money and its implications for security valuation. The mechanics of continuously compounded interest rates is provided first, and then the mechanics are applied to security valuation. The third section focuses on the valuation of, perhaps, the simplest type of security—a discount bond. The value of a discount bond is simply the present value of its promised payment at maturity. The fourth section focuses on coupon bonds and shows that they are simply portfolios of discount bonds. In both sections, the valuation formulas are used to develop the interest rate risk measures of duration and convexity. Since coupon bonds have multiple cash flows through time, the fifth section addresses the issue of maturity-specific interest rates. Zero-coupon interest rates are shown to imply forward rates of interest. Finally, the interest rate mechanics are applied to common stock valuation. The value of a share of stock is shown to be the present value of an infinite series of expected dividend payments.

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APPENDIX 2A: TAYLOR SERIES EXPANSION OF BOND VALUE

In using the duration and convexity to predict bond price movements, we implicitly used a *Taylor series expansion* of the bond valuation formula. From calculus, we know that most smooth functions $f(x)$ can be expanded in a *Taylor series* about the point x_0,[16] that is,

$$\begin{aligned} f(x) &= f(x_0) + \frac{f'(x_0)}{1}(x - x_0) + \frac{f''(x_0)}{1 \cdot 2}(x - x_0)^2 + \frac{f'''(x_0)}{1 \cdot 2 \cdot 3}(x - x_0)^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \end{aligned} \tag{2A.1}$$

In (2A.1), replace x with the yield to maturity of the bond, r, and $f(x)$ with the bond valuation function, $B(r)$.

APPENDIX 2B: SUM OF A GEOMETRIC PROGRESSION

A *geometric progression* is a sequence of numbers, a_i, $i = 1, \ldots, n$, whose adjacent terms satisfy the property that

$$\frac{a_{i+1}}{a_i} = b$$

where b is called the *common ratio*. The sum of an n element geometric series whose first element is 1 is

$$S_n = 1 + b + b^2 + \cdots + b^{n-3} + b^{n-2} + b^{n-1} \tag{2B.1}$$

While this sum may seem tedious to compute, it may be simplified considerably. First, multiply (2B.1) by b.

$$bS_n = b + b^2 + \cdots + b^{n-2} + b^{n-1} + b^n \tag{2B.2}$$

Now, subtract (2B.2) from (2B.1).

$$(1 - b)S_n = 1 - b^n$$

[16] For the special case where $x_0 = 0$, (A.1) is sometimes called the Maclaurin series of $f(x)$.

or

$$S_n = \frac{1 - b^n}{1 - b} \tag{2B.3}$$

Where the number of elements in the series is infinite (i.e., $n = \infty$) and $b < 1$, the sum of the geometric progression is

$$S_\infty = \frac{1}{1 - b} \tag{2B.4}$$

If the infinite series begins with b, the sum is

$$S_\infty = \frac{b}{1 - b} = \frac{1}{1/b - 1} \tag{2B.5}$$