PORTFOLIO THEORY

The expected utility of wealth function is helpful in many decision-making contexts. However, a weakness of the framework is that we need to specify the individual's utility function. Exactly how one identifies the mathematical structure of an individual's utility function is unclear. Fortunately, a specific structure is unnecessary for the individual's portfolio allocation decision. The reason is that individuals with diminishing positive marginal utility of wealth (i.e., so-called risk-averters) have implied indifference curves that express expected return-risk preferences.

Indifference curves

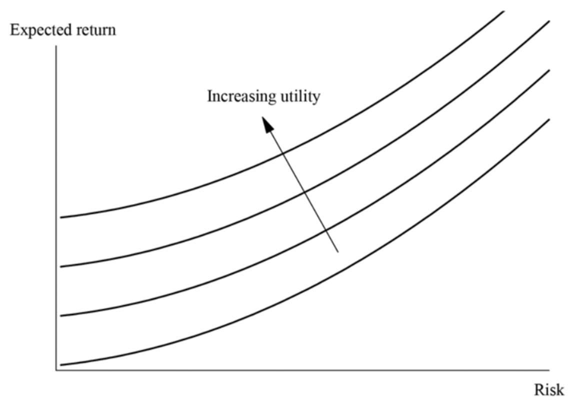
More specifically, Tobin (1958) shows that individuals with diminishing positive marginal utility have expected return (E) / risk (σ) indifference curves shaped like those shown in Figure 1(a),[1] where the standard deviation of return measures risk. Along each indifference curve, the expected utility is held constant. The curves have the properties $dE/d\sigma > 0$ and $d^2E/d\sigma^2 > 0$. The first derivative says an individual will demand a higher return as risk increases. The second derivative says that the rate at which the individual requires more return grows faster and faster as risk increases. In Figure 1(a), note also that the higher the indifference curve, the greater the expected utility. That means individuals choose portfolios with the highest expected return for a given level of risk. Such portfolios are called *efficient portfolios*.

Before formulating the individual's portfolio allocation decision, it is worthwhile to note how a risk-averter's indifference curves differ from those of a risk-neutral individual. Figure 1(b) illustrates the indifference curves of a risk-neutral individual. The fact that the curves are horizontal means a risk-neutral individual does not care about risk. Such an individual chooses a portfolio that maximizes expected return. Vertical indifference curves are at the other behavioral extreme, as shown in Figure 1(c). This individual is a *risk-minimizer* and will choose a portfolio that minimizes portfolio risk.

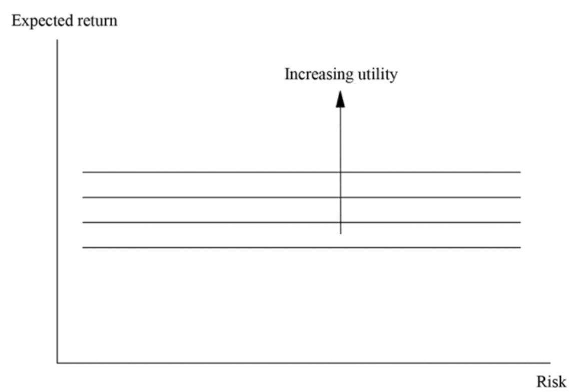
[1] Technically speaking, Tobin (1958) proved this result in two general cases: (a) individuals have quadratic utility of wealth, and (b) the distribution of security returns is multivariate normal.

Figure 1: Indifference curves of individuals who are (a) risk-averse, (b) risk-neutral, and (c) risk-minimizing.

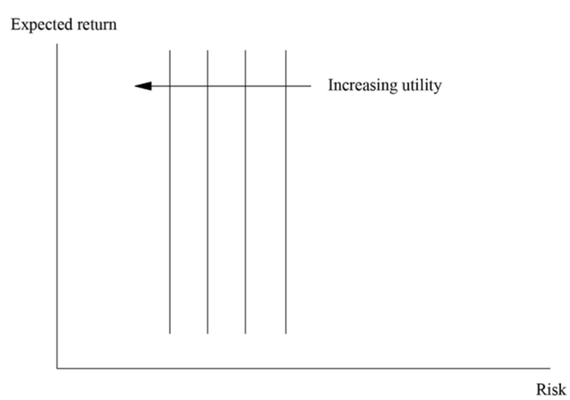
(a) Risk-averse



(b) Risk-neutral



(c) Risk-minimizing



Portfolio decision-making

The focus now turns to identifying efficient portfolios, that is, portfolios with the highest expected return for a given level of risk. To do so, an individual must gather a considerable amount of information. Assuming n risky securities exist in the marketplace, an individual must estimate (a) the expected return of each risky security, E_i, $i = 1, \cdots, n$, (b) the standard deviation of return of each risky security, σ_i, $i = 1, \cdots, n$, and (c) the correlation of returns for each pair of securities in the marketplace, ρ_{ij}, $i = 1, \cdots, n$ and $j = 1, \cdots, n$. At first blush, there is a need to estimate n^2 different correlation coefficients. Concerning these correlations, however, we know that $\rho_{ij} = +1$ where $i = j$ and that $\rho_{ij} = \rho_{ji}$. The number of necessary estimates $n(n-1)/2$ is. For expositional convenience, covariances are used below. The covariance between the returns of securities *i* and *j* is defined as $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$.

Specific definitions are required to set up the portfolio allocation problem. The expected return on portfolio *S* is

$$E_S = \sum_{i=1}^{n} X_i E_i , \tag{1}$$

where X_i is the proportion of the individual's wealth invested in security *i*. Naturally, the sum of the proportions equals 1, that is, $\sum_{i=1}^{n} X_i = 1$. This is sometimes called the *wealth constraint*. The standard deviation of the portfolio return is

$$\sigma_S = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j \sigma_{ij}} . \tag{2}$$

Now, to identify the individual's optimal allocation among the n risky securities, we minimize portfolio risk,

$$\text{Minimize } \sigma_S^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j \sigma_{ij} , \tag{3}$$

subject to:

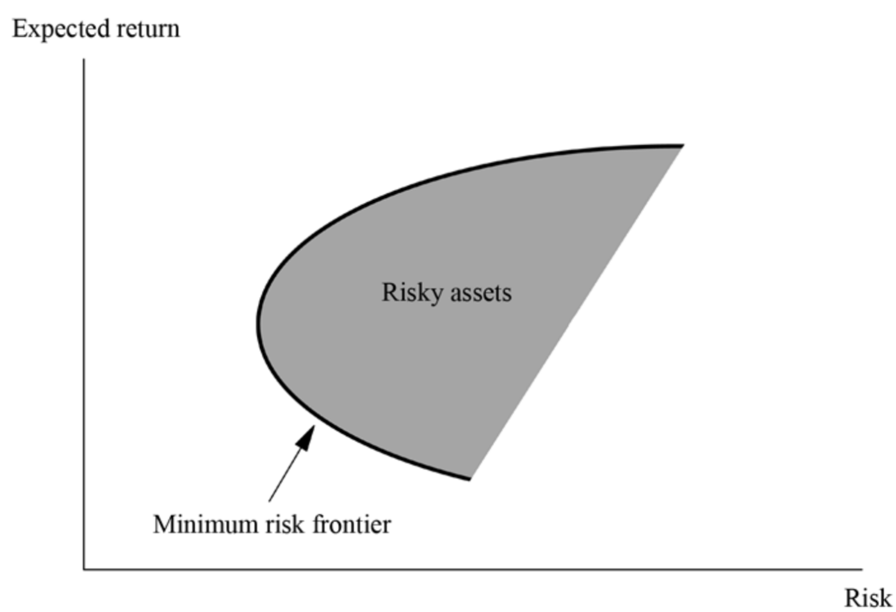
$$\sum_{i=1}^{n} X_i E_i = E_S \tag{3a}$$

and

$$\sum_{i=1}^{n} X_i = 1 . \tag{3b}$$

Constraint (3a) requires that the weights produce an expected portfolio return equal to the target level, E_S, and constraint (3b) requires that all risky security wealth is fully allocated. The objective function (3), together with the constraints (3a) and (3b), constitute a *nonlinear programming problem*. Some such problems can be solved analytically; others numerically. For current purposes, however, it is sufficient to know that, as long as no two risky securities have perfectly correlated returns, the solution to the problem is a unique set of allocations, X_i^*, $i = 1, \cdots, n$, that produce a *minimum variance portfolio*. If we solve this portfolio allocation problem for a range of target expected portfolio return E_S levels, we can trace the minimum variance (or minimum risk) frontier shown in Figure 3. This frontier is sometimes referred to as the *Markowitz (1952) frontier*, in honor of Nobel Laureate Harry Markowitz, who originally developed the framework more than 70 years ago.

Figure 3: Minimum variance (or Markowitz) frontier



<u>Illustration 1</u>: Identify efficient portfolios using two risky securities.

Describe the range of efficient portfolio allocations when only two risky securities are available in the marketplace. The expected returns and standard deviations of returns of the two securities are shown below. Assume the correlation between security 1 and security 2 returns is .25.

Security	Expected return	Standard deviation
1	18%	20%
2	12%	16%

You must first identify the feasible portfolios to identify the set of efficient portfolios. The expected return and standard deviation of return of portfolios created by allocating wealth between security 1 and security 2 are given by (1) and (2), where the number of securities n equals 2. Thus, the expected portfolio return is

$$\begin{aligned} E_S &= X_1 E_1 + (1 - X_1) E_2 \\ &= X_1(.18) + (1 - X_1)(.12), \end{aligned}$$

and the standard deviation of portfolio return is

$$\begin{aligned} \sigma_S &= \sqrt{X_1^2 \sigma_1^2 + 2X_1(1 - X_1)\rho_{12}\sigma_1\sigma_2 + (1 - X_1)^2 \sigma_2^2} \\ &= \sqrt{X_1^2(.20)^2 + 2X_1(1 - X_1)(.25)(.20)(.16) + (1 - X_1)^2(.16)^2} \end{aligned}$$

Since only two securities exist, the proportion of wealth invested in security 2 is $X_2 = 1 - X_1$. The rest of the exercise is a matter of computing E_S σ_S for different levels X_1. Excel is a useful computational tool. See Sheet 1 of Portfolio theory illustrations.xlsx.

Proportion of wealth invested in security		Portfolio attributes	
		Expected	Standard
1	2	return	deviation
1.0	0.0	18.00%	20.00%
0.9	0.1	17.40%	18.47%
0.8	0.2	16.80%	17.08%
0.7	0.3	16.20%	15.89%
0.6	0.4	15.60%	14.95%
0.5	0.5	15.00%	14.28%
0.4	0.6	14.40%	13.95%
0.355	0.645	14.13%	13.91%
0.3	0.7	13.80%	13.97%
0.2	0.8	13.20%	14.33%
0.1	0.9	12.60%	15.03%
0.0	1.0	12.00%	16.00%

The above table shows that the expected portfolio return falls from 18% to 12% as the proportion of wealth invested in security 1 goes from 1 to 0. On the other hand, the standard deviation of portfolio return initially falls as X_1 it is reduced but then rises again after X_1 passing the level of .40 on its way to zero. The figure below summarizes the results. Exactly what allocation produces the minimum risk portfolio can be determined by taking the derivative of the portfolio standard deviation and setting it equal to 0. For the two-security portfolio, the minimum risk allocation is

$$X_1 = \frac{\sigma_2^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}. \qquad (4)$$

Substituting the problem parameters, we find that the risk-minimizing portfolio is created by allocating .355 wealth to security 1 and .645 to security 2. This portfolio has an expected return of 14.13% and a standard deviation of 13.91%. Thus, while the above table shows the range of feasible portfolios that can be created by allocating one's wealth between security 1 and security 2, no risk-averse individual will hold a portfolio with less (more) than .355 (.645) of his wealth allocated to security 1 (2). The range of allocations that produces *efficient portfolios* is $.355 \le X_1 \le 1$.[2]

A short digression is helpful here. While we have identified the range of allocations that produces efficient portfolios, one efficient portfolio—the minimum risk portfolio—will never be held by a risk-averter. The reason is that the slope of the expected return/risk frontier at the minimum risk portfolio is infinite—an individual whose indifference curves have the properties $dE/d\sigma > 0$ and $d^2E/d\sigma^2 > 0$ is not allowed to choose such a portfolio. Consequently, the range of portfolios from which a risk-averter selects his *optimal portfolio* from the set of *efficient portfolios* is defined $.355 < X_1 \le 1$.

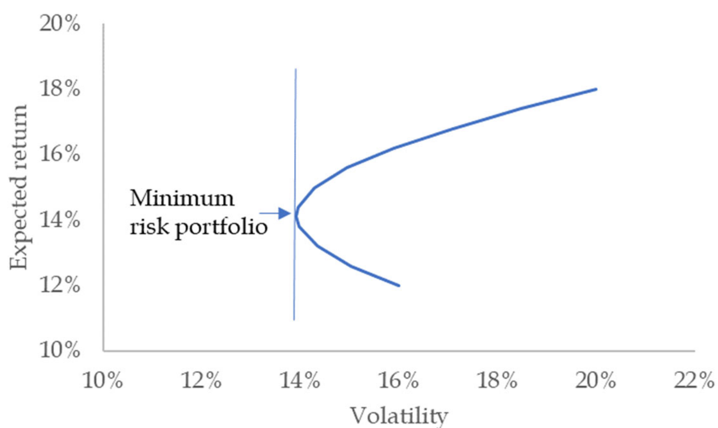


***Illustration 2*: Identify efficient portfolios using traditional asset classes.**

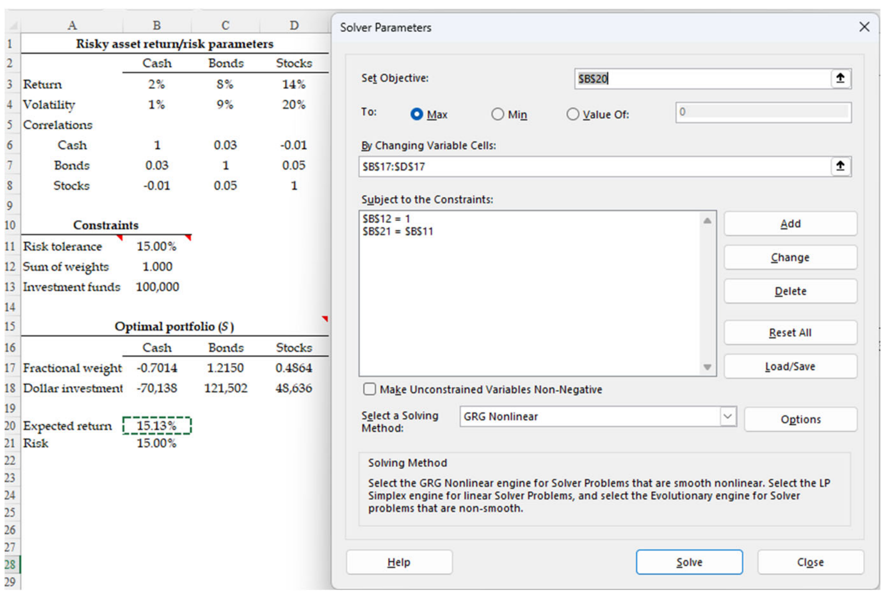
The traditional asset classes are stocks, bonds, and cash equivalents. Based upon your analysis of historical data and the economic outlook, you have developed estimates of expected asset category returns, standard deviations of returns, and

[2] Technically speaking, the range of efficient portfolios continues to the right of security 1 since short sales of security 2 are permitted.

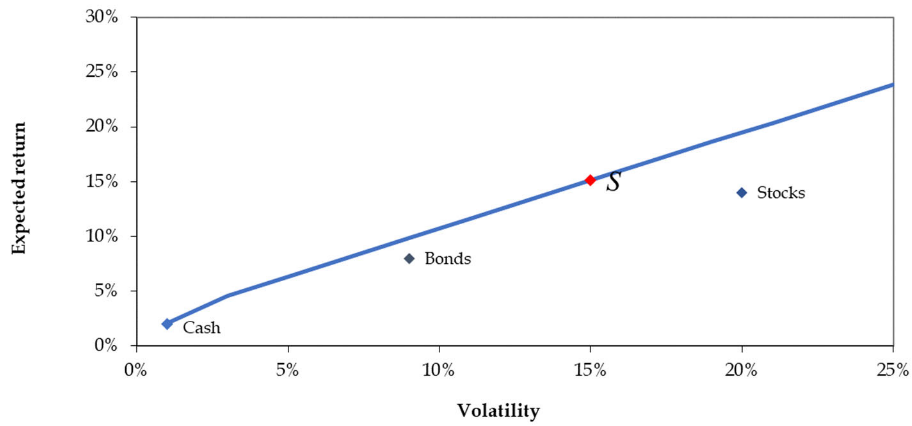
correlations between return pairs. Find the risk-minimizing portfolio with a target expected return of 15%.

Risky asset return/risk parameters			
	Cash	Bonds	Stocks
Return	2%	8%	14%
Volatility	1%	9%	20%
Correlations			
Cash	1	0.03	-0.01
Bonds	0.03	1	0.05
Stocks	-0.01	0.05	1

Solving this problem analytically is possible but cumbersome. Excel's Solver is a convenient means (i.e., a numerical search procedure) for finding the solution. Refer to Sheet 2 of Portfolio theory illustrations.xlsx. With a risk tolerance of 15% the optimal allocations are -0.7014 in cash, 1.2150 in bonds, and 0.4564 in stocks. The portfolio's expected return is 15.13%.



We repeat the exercise with a range of target returns to generate the efficient frontier. The table and figures below show the results. As risk tolerance rises, funds are taken from cash equivalents (i.e., a relatively risk-free asset) and invested in risky bonds and stocks.



Expected return	Risk tolerance	Optimal portfolio weights		
		Cash	Bonds	Stocks
2.09%	1.00%	0.9886	0.0086	0.0028
4.55%	3.00%	0.6691	0.2366	0.0943
6.35%	5.00%	0.4352	0.4038	0.1610
8.12%	7.00%	0.2060	0.5677	0.2264
9.88%	9.00%	-0.0213	0.7294	0.2919
11.63%	11.00%	-0.2483	0.8914	0.3568
13.38%	13.00%	-0.4749	1.0532	0.4217
15.13%	15.00%	-0.7014	1.2150	0.4864
16.87%	17.00%	-0.9272	1.3756	0.5516
18.62%	19.00%	-1.1540	1.5384	0.6156
20.36%	21.00%	-1.3804	1.7004	0.6800
22.11%	23.00%	-1.6057	1.8601	0.7455
23.85%	25.00%	-1.8318	2.0217	0.8101

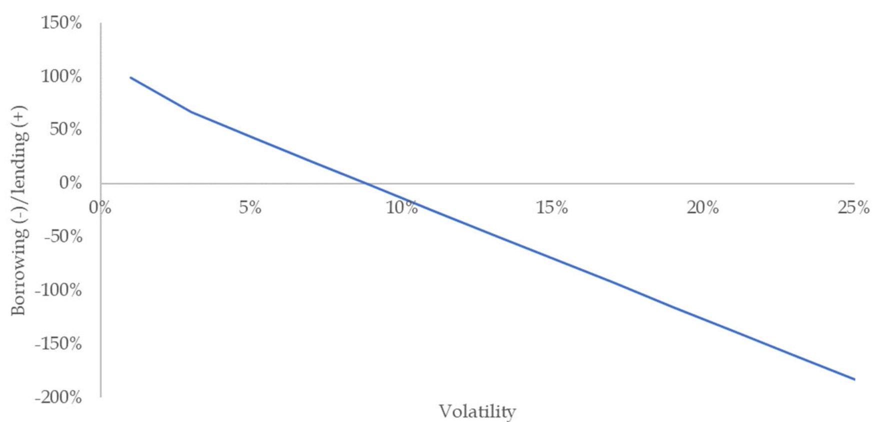
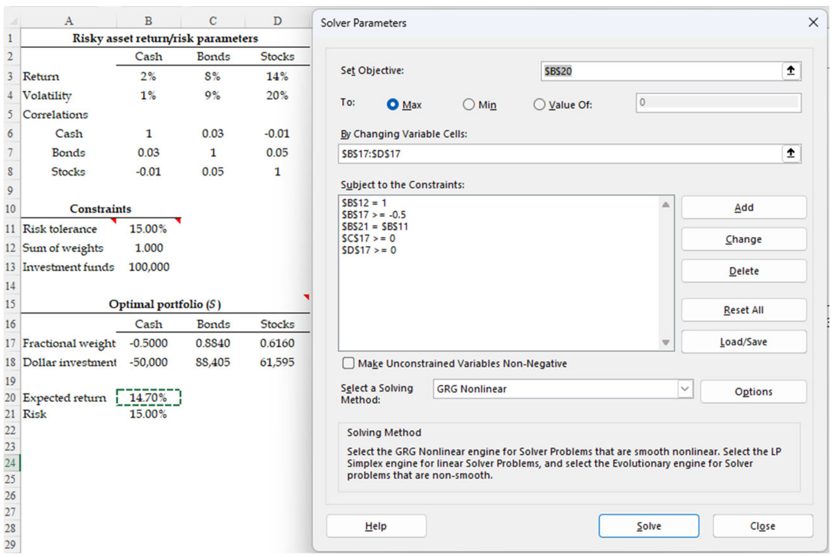


***Illustration 3*: Identify efficient portfolios using borrowing and short sales constraints.**

Often, finding the optimal portfolio involves imposing practical constraints. Short sales of risky asset categories like stocks and bonds are usually prohibited. Short sales of cash equivalents (i.e., borrowing) are allowed but often constrained. The SEC's Regulation T limits borrowing to 50% of the overall value of the risky assets (i.e., the allocation to cash equivalents cannot be less than -0.50. Re-do the allocations in Illustration 1, imposing these additional constraints.

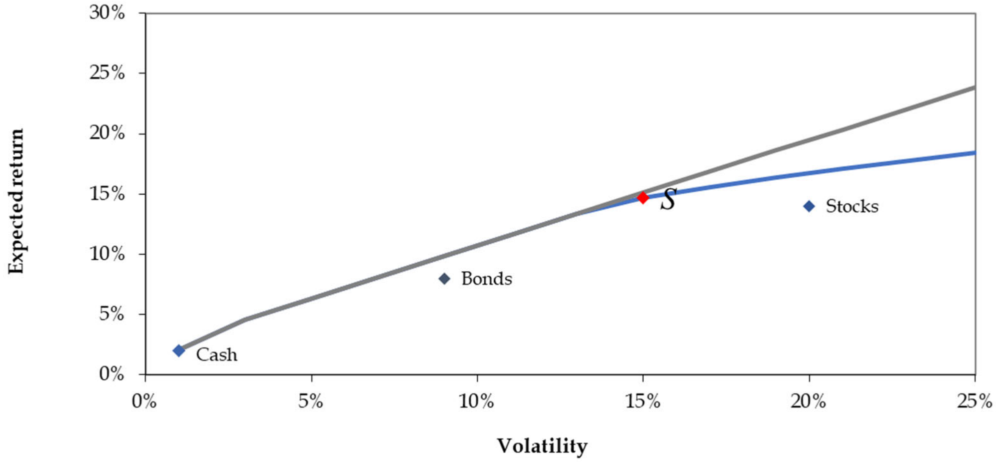
The constraints are written into Solver as follows:



The consequences are intuitive, as shown in the table below. At low-risk tolerance levels, the allocations do not change from the previous illustration. With higher risk, the borrowing constraint becomes binding.

Expected return	Risk tolerance	Optimal portfolio weights		
		Cash	Bonds	Stocks
2.09%	1.00%	0.9886	0.0086	0.0028
4.55%	3.00%	0.6691	0.2366	0.0943
6.35%	5.00%	0.4352	0.4038	0.1610
8.12%	7.00%	0.2060	0.5677	0.2264
9.88%	9.00%	-0.0213	0.7294	0.2919
11.63%	11.00%	-0.2483	0.8914	0.3568
13.38%	13.00%	-0.4749	1.0533	0.4216
14.70%	15.00%	-0.5000	0.8840	0.6160
15.60%	17.00%	-0.5000	0.7333	0.7667
16.38%	19.00%	-0.5000	0.6035	0.8965
17.10%	21.00%	-0.5000	0.4841	1.0159
17.77%	23.00%	-0.5000	0.3709	1.1291
18.43%	25.00%	-0.5000	0.2619	1.2381

Shown differently, we combine the efficient frontiers from this illustration and the last.



<u>Illustration 4</u>: Identify efficient portfolios when allocations have maximum limits.

In addition to borrowing and short sales constraints, investment managers often have policies preventing them from exceeding certain maximum limits. The policies may say, for example, that at most 50% of the fund can be in bonds and at most 80% can be in stocks. Impose these limits and explain the new results.

Naturally, imposing further constraints cannot enhance the return/risk opportunities. The allocations are now:

Expected	Risk	Optimal portfolio weights		
return	tolerance	Cash	Bonds	Stocks
2.09%	1.00%	0.9886	0.0086	0.0028
4.55%	3.00%	0.6691	0.2366	0.0943
6.35%	5.00%	0.4352	0.4038	0.1610
8.12%	7.00%	0.2064	0.5668	0.2268
9.76%	9.00%	0.0533	0.6000	0.3467
11.19%	11.00%	-0.0659	0.6000	0.4659
12.53%	13.00%	-0.1779	0.6000	0.5779
13.83%	15.00%	-0.2862	0.6000	0.6862
15.11%	17.00%	-0.3924	0.6000	0.7924
15.20%	17.15%	-0.4000	0.6000	0.8000
15.20%	17.15%	-0.4000	0.6000	0.8000
15.20%	17.15%	-0.4000	0.6000	0.8000
15.20%	17.15%	-0.4000	0.6000	0.8000

The effects are felt most strongly in the table and figure below at high-risk tolerance levels. Indeed, portfolios with volatilities exceeding 17.15% are not feasible. At 17.15%, the maximum allocations for bonds and stocks are binding. Only 40% of portfolio value is needed to reach these maximum allocations, so the borrowing constraint is not binding.

REFERENCES AND SUGGESTED READINGS

Markowitz, Harry, 1952, Portfolio selection, *Journal of Finance* 12 (March), 77-91.

Tobin, James, 1958, Liquidity preference as behavior towards risk, *Review of Economic Studies* 25 (February), 65-86.