UTILITY THEORY

In most financial economic models, individuals are assumed to be *risk-averse*. Investors do not like risk but are willing to bear it if paid an adequate risk premium. Risk premiums arise from how an individual's satisfaction varies with wealth. A *utility of wealth function* $U(w)$ is the level of satisfaction (measured in units of utility) realized from having a level of wealth. An individual's marginal utility of wealth is assumed to be positive (i.e., $dU(w)/dw > 0$)—the more wealth, the more satisfaction. Indeed, this property is the driving force behind the absence of costless arbitrage opportunities in a rationally functioning marketplace. As wealth increases, however, the rate at which satisfaction increases falls (i.e., $d^2U(w)/dw^2 < 0$). The next dollar earned is not quite as satisfying as the last.

Figure 1 illustrates the shape of the utility function for an individual with diminishing positive marginal utility of wealth. Note that as wealth increases, utility increases, but at a decreasing rate. Consider his behavior when presented with a fair bet to show that this individual is a risk-averter. A *fair bet* is any bet whose expected outcome is 0. For example, a 50-50 chance of winning or losing X constitutes a fair bet. Accepting a fair bet implies no change in the individual's expected wealth level. If the individual's certain wealth level before the bet is w_0, his expected wealth level upon accepting the bet remains at w_0, that is,

$$E(\tilde{w}) = .5\left(w_0 + \tilde{X}\right) + .5\left(w_0 - \tilde{X}\right) = w_0 .$$

But, because the individual's expected wealth does not change, that does not mean he is indifferent about whether to accept the bet. He will not. The reason is that, after taking the bet, his expected satisfaction level is

$$E\left[\tilde{U}(w)\right] = .5U\left(w_0 + \tilde{X}\right) + .5U\left(w_0 - \tilde{X}\right).$$

Because his utility function is concave from below, as shown in Figure 2, the expected utility of wealth after taking the bet rests below the utility that he had to begin with, $E\left[\tilde{U}(w)\right] < U(w_0)$. Because taking a fair bet reduces the individual's expected utility, an individual with a diminishing positive marginal utility of wealth is said to be a *risk-averter*.[1]

[1] In contrast, an individual with *constant positive* marginal utility of wealth is called *risk-neutral* and will be indifferent to a fair bet. An individual with *increasing positive* marginal utility of wealth is called *risk-loving* and will pay for a fair bet. An individual with *a negative* marginal utility of wealth is irrational.

Figure 1: Utility function of a risk-averse individual.

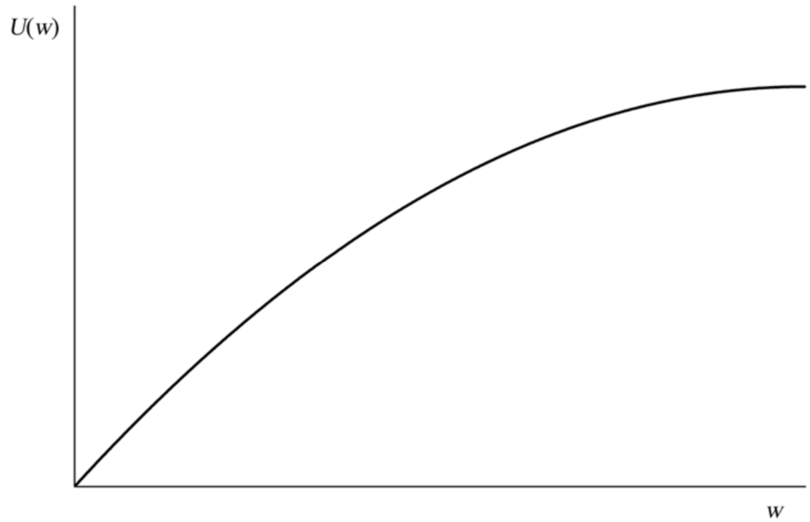
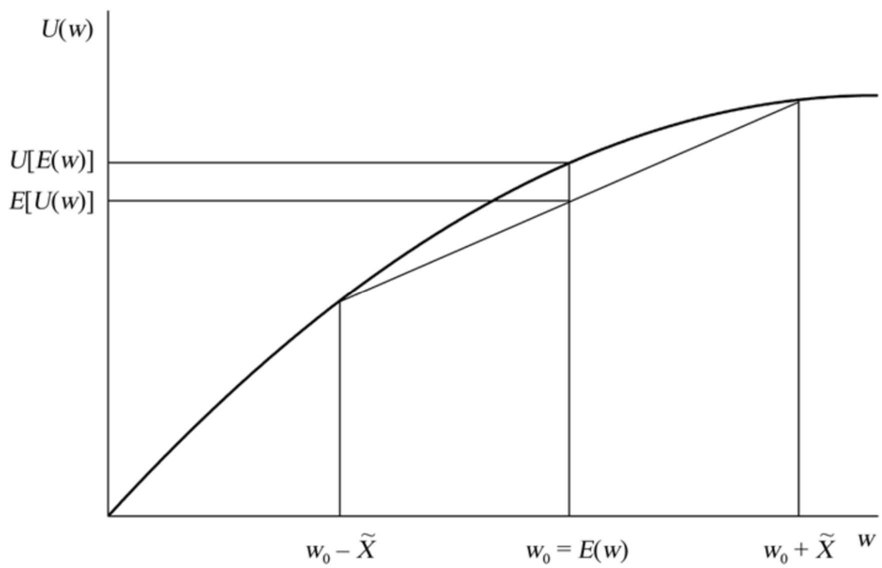


Figure 2: Utility function of a risk-averse individual when evaluating a fair bet.



Utility theory is functional not only in demonstrating general behavioral principles but also in evaluating specific investment opportunities. To make particular decisions, however, the mathematical character of the utility function needs to be more precisely defined.[2] A logarithmic utility of wealth function,

[2] The four most commonly used utility of wealth functions used in financial economics are (a) the logarithmic utility function, $U\left(w\right) = \ln w$, (b) the quadratic utility function, $U\left(w\right) = aw - bw^2$

$U(w) = \ln w$, mimics the behavior of a risk-averter— $dU(w)/dw = 1/w > 0$ and $d^2U(w)/dw^2 = -1/w^2 < 0$. So does a square root utility of wealth function, $U(w) = \sqrt{w}$, since $dU(w)/dw = .5w^{-.5} > 0$ and $d^2U(w)/dw^2 = -.25w^{-1.5} < 0$. We use these utility functions in the illustrations that follow.

Aside from knowing the specific character of the utility function, a handful of definitions are also important. In the following illustrations, we assume that the individual holds two assets—a risk-free asset whose value is R and a risky asset whose value in one period is either $\tilde{X}_1$ or $\tilde{X}_2$, with probabilities p and $1-p$, respectively. Assuming the risk-free interest rate is zero, the individual's *expected utility of terminal wealth* is

$$E\left[\tilde{U}(w)\right] = pU\left(R + \tilde{X}_1\right) + (1-p)U\left(R + \tilde{X}_2\right). \tag{1}$$

Suppose someone approaches this individual and asks him to sell his risky asset. What is the least amount that the individual will take? We must first identify the cash equivalent of the individual's overall position to answer this question. The *cash equivalent* is that certain amount of cash, C, that the individual is willing to take for his entire position and is computed by setting the utility of the cash amount equal to the expected utility of terminal wealth, that is,

$$U(C) = E\left[\tilde{U}(w)\right]. \tag{2}$$

With the amount of the cash equivalent, C, known, the *minimum selling price* of the risky asset equals $C - R$.

Illustration 1: Identify maximum insurance premium.

Consider two individuals – A with a logarithmic utility function and B with a square root utility function. Both individuals have \$100,000 in cash and face losing \$50,000, with a 5% probability. What is the maximum amount each individual would be willing to pay for insurance?

Individual A currently enjoys an expected satisfaction level,

$$\begin{aligned} E\left[\tilde{U}_A(w)\right] &= .05\ln(100{,}000 - 50{,}000) + .95\ln(100{,}000) \\ &= 11.478. \end{aligned}$$

Holding expected utility constant, this implies that A is indifferent between staying in his current position (i.e., keeping \$100,000 in cash and having the

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$a > 2bw$ and $b > 0$, (c) the exponential utility function, $U(w) = -e^{-aw}$ where $a > 0$, and (d) the power utility function $U(w) = w^a$ where $0 < a < 1$.

prospect of losing $50,000) and having a certain amount of cash *C* as determined by

$$U_A(C) = \ln C = 11.478 .$$

Solving for *C*, you find $C = e^{11.478} = 96,593.63$. In other words, A is indifferent between having (a) $100,000 in cash and running a 5% chance of losing $50,000 and (b) $96,593.63 in cash. Thus, the maximum amount A will pay for insurance against loss is $100,000-96,593.63 or $3,406.37.

Individual B currently enjoys a satisfaction level,

$$E\left[\tilde{U}_B(w)\right] = .05\sqrt{100,000-50,000} + .95\sqrt{100,000}$$
$$= 311.597.$$

Individual B's cash equivalent wealth level is determined by

$$U_B(C) = \sqrt{C} = 311.597 .$$

Solving for *C*, $C = 311.597^2 = 97,092.51$, which means B is willing to pay up to $2,907.49 for insurance against loss. An individual with logarithmic utility is more risk averse than an individual with square root utility due to the slope and the curvature of the two functions.

***Illustration 2**: Are derivatives contract markets a zero-sum game?*

Derivative trades are zero-sum games – what the buyer gains, the seller loses, and vice versa. However, because the trade outcomes are zero-sum, it does not imply that both the buyer and the seller can benefit from trading. Assume Individual A has $50 in cash and one share of stock. The stock, he believes, has a 60% chance of falling in price to $80 and a 40% chance of increasing in price to $120. Individual B has $100 in cash and no other holdings. B, however, follows the stock held by A and is much more optimistic regarding its prospects. Specifically, B assigns only a 30% chance of the stock falling in price to $80 and a 70% chance of it increasing to $120. Demonstrate that A and B can be better off by trading a put option written on the stock. Assume the put has an exercise price of $100 and costs $10. Assume both individuals have square root utility functions and ignore the time value of money.

Assume Individual A wants to buy the put, considering his pessimistic outlook regarding the stock's prospects. A's current expected utility level is

$$E[\tilde{U}(w_A)] = .6\sqrt{50+80} + .4\sqrt{50+120} = 12.06 ,$$

where the terminal wealth levels are 130 or 170, depending on the stock's performance; on the other hand, if he buys the put, his terminal wealth level is 130 less the put price plus the payoff if the put price falls and is 170 less the put price if the stock price rises. Thus, if he buys the put, his expected utility is

$$E\left[\tilde{U}(w_A)\right] = .6\sqrt{130 - 10 + (100 - 80)} \\ +.4\sqrt{170 - 10 + 0} = 12.16 .$$

Thus, from an expected utility of terminal wealth standpoint, A is better off buying the put.

On the other hand, Individual B is more optimistic regarding the stock's prospects and is considering selling the put. B currently enjoys a utility of wealth equal to

$$U(w_B) = \sqrt{100} = 10 .$$

If he sells the put, his terminal wealth will be either 100 plus the put price less the put payoff (that goes to A) if the stock price falls or 100 plus the put price if the price rises. Thus, after selling the put, his expected utility of wealth is

$$E\left[\tilde{U}(w_B)\right] = .3\sqrt{100 + 10 - (100 - 80)} \\ +.7\sqrt{100 + 10 - 0} = 10.09.$$

Since selling the put provides B with a portfolio with a higher expected utility of wealth, the trade also makes B better off.

The fact that both A and B are made better off by trading with each other does not negate the fact that the trade is zero-sum. It is. If the stock price falls, A's net payoff equals the option payoff less the put price, that is, $(100 - 80) - 10 = +10$, and B's net payoff is the put price less the exercise proceeds $10 - (100 - 80) = -10$. If the stock price rises, the put expires worthless, which means that A's net loss on the put, 10, is B's net gain.