Option valuation

Analytical formulas

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Option valuation

- Purpose:
  - Develop European-style option valuation formula.
    - Called "Black-Scholes" or "Black-Scholes/Merton" (BSM) call option valuation model.

$$c = Se^{-iT}N(d_1) - Xe^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln(S/X) + (r - i + .5\sigma^2)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

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Option valuation

- Three theoretical approaches to option valuation:
  - Risk-free hedge portfolio–Black/Scholes (1973)
    - Most well-known
  - Risk-neutral investors–Cox/Ross (1976)
    - Most intuitive
  - Risk-averse investors–Samuelson (1965)
    - Had formula 8 years earlier

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Option valuation

- Understanding all three approaches will:
  - Provide economic intuition underlying option valuation.
  - Identify situations in which models do not apply.

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Option valuation

- Use simple binomial model to illustrate three approaches.
  - Will show all produce same option value.

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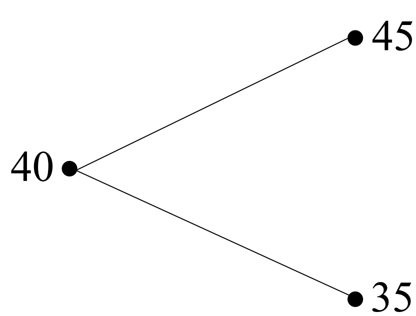
1. Risk-free hedge portfolio

- Suppose current asset price is 40 and price in 3 months will be 45 or 35.



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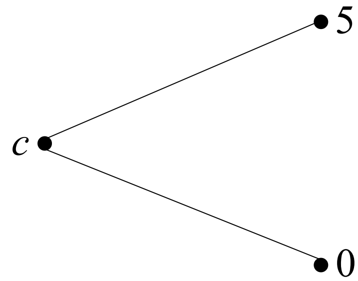
1. Risk-free hedge portfolio

- Suppose you want to value 3-month European call with 40 exercise price.
  - Call values at expiration are known.



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1. Risk-free hedge portfolio

- Set terminal values equal to one another (thereby creating risk-free hedge) and solve for *n*.

$$45 - 5n = 35$$

$$n = 2$$

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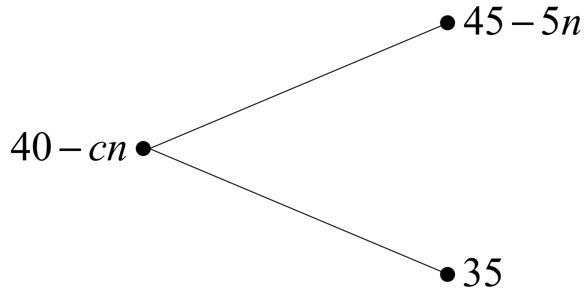
1. Risk-free hedge portfolio

- Form portfolio by buying 1 unit of asset and selling *n* 3-month call options at price *c*.



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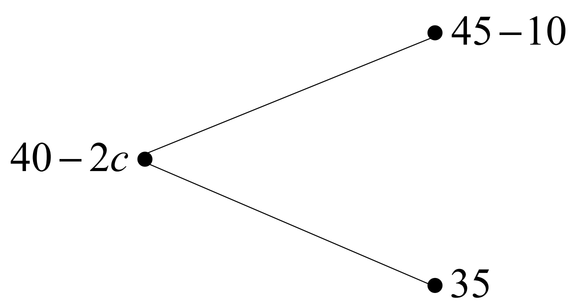
1. Risk-free hedge portfolio

- Substitute into binomial lattice.



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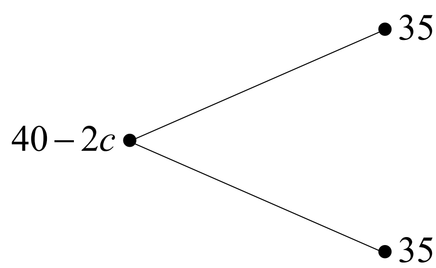
1. Risk-free hedge portfolio

- Portfolio is risk-free because its terminal value (i.e., 35) is invariant to asset price.



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1. Risk-free hedge portfolio

- 40-2*c* invested in T-bills would also have terminal value of 35.
- If risk-free rate over 3-month interval is 2%, no-arbitrage implies

$$40 - 2c = \frac{35}{1.02}$$
$$c = 2.84$$

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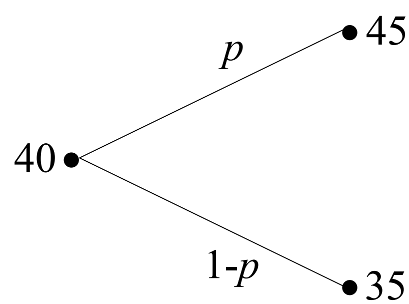
1. Risk-free hedge portfolio

- Risk-free hedge portfolio was formed without knowing probabilities.



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1. Risk-free hedge portfolio

- Risk-free hedge portfolio was formed without knowing probabilities.
- Implies call value was derived without knowing investor risk/return preferences.

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2. Risk-neutral valuation

- Since investor preferences are not required to value call, assume investors are risk-neutral.
  - Under risk-neutrality, all assets have expected return equal to risk-free interest rate.

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2. Risk-neutral valuation

- To value call under risk-neutrality, need to specify upstate and downstate probabilities.
  - To identify implied risk-neutral probabilities, equate expressions for terminal asset price and solve for *p*.

$$40(1.02) = 45p + 35(1-p)$$

$$p = 58\%$$

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2. Risk-neutral valuation

- With risk-neutral probabilities known, value call using traditional approach (i.e., take PV of expected future value).

$$c = \frac{E(\tilde{c}_T)}{1 + r^*}$$

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2. Risk-neutral valuation

- With risk-neutral probabilities known, value call using traditional approach (i.e., take PV of expected future value).

$$E(\tilde{c}_T) = 5(.58) + 0(.42) = 2.90$$

$$c = \frac{2.90}{1.02} = 2.84$$

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3. Risk-averse valuation

- If investors are risk-averse, expected return on risky asset will be higher than risk-free interest rate.
- Implies probability of an upstate is higher under risk aversion.

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3. Risk-averse valuation

- Assume expected asset return is 4%. i.e., has 2% risk premium.

$$40(1.04) = 45p' + 35(1 - p')$$
$$p' = 66\%$$

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3. Risk-averse valuation

- With risk-averse probabilities known, value call using traditional approach.

$$c = \frac{E(\tilde{c}_T)}{1 + E_c^*}$$

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3. Risk-averse valuation

- What is expected terminal call value?

$$E(\tilde{c}_T) = 5(.66) + 0(.34) = 3.30$$

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3. Risk-averse valuation

- What is required rate of return on call?
  - Know only risk-free rate

$$E_c^* = .02 + (E_M^* - .02)\beta_c$$

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3. Risk-averse valuation

- What is required rate of return on call?
  - CAPM says call's expected return is:

Risk-free rate of interest

Systematic risk of call

$$E_c^* = .02 + (E_M^* - .02)\beta_c$$

Expected market return

Expected return on call

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3. Risk-averse valuation

- What is required rate of return on call?
  - Call's beta equals asset's beta times elasticity of call price with respect to asset price.

$$E_c^* = .02 + (E_M^* - .02)\beta_S \left(\frac{dc / c}{dS / S} \right)$$

$$\downarrow$$

$$E_c^* = .02 + (E_M^* - .02)\beta_S \left(\frac{dc}{dS} \times \frac{S}{c} \right)$$

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3. Risk-averse valuation

- What is required rate of return on call?
  - Call's beta equals asset's beta times elasticity of call price with respect to asset price.

$$E_c^* = .02 + (E_M^* - .02)\beta_S \left(\frac{5-0}{45-35} \times \frac{40}{c} \right)$$

$$= .02 + (E_M^* - .02)\beta_S \left(\frac{20}{c} \right)$$

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3. Risk-averse valuation

- What is required rate of return on call?
  - Already know:

$$E_S^* = .04 = .02 + (E_M^* - .02)\beta_S$$
$$\Rightarrow (E_M^* - .02)\beta_S = .02$$

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3. Risk-averse valuation

- What is required rate of return on call?
  - Hence,

$$E_c^* = .02 + .02\left(\frac{20}{c}\right)$$

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3. Risk-averse valuation

- Value of call is therefore:

$$c = \frac{3.30}{1 + .02 + .02\left(\frac{20}{c}\right)}$$

$$1.02c + .40 = 3.30$$

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3. Risk-averse valuation

- Value of call is therefore:

$$c = \frac{3.30}{1 + .02 + .02\left(\frac{20}{c}\right)}$$

$$1.02c + .40 = 3.30$$

$$c = \frac{3.30 - .40}{1.02} = 2.84$$

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3. Risk-averse valuation

- Supporting file: Simple binomial valuation.xlsx

Simple binomial option valuation

Asset		Initial price	Probability	Terminal price
Price	40		0.660	45
Price increment	5	40		
Risk-free interest rate	2.00%		0.340	35
Expected risk premium	2.00%			
Expected return	4.00%			
Expected terminal price	41.60			

Call		Initial price	Probability	Terminal price
Exercise price	40		0.660	5
Expected terminal price	3.30	c		
Risk adjustment factor	20.00		0.340	0
Current value	2.84			
Expected return	16.07%			
Interest rate	2.00%			

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Option valuation

- All three approaches lead to same result.
  - Risk-free hedge portfolio valuation
  - Risk-neutral valuation
  - Risk-averse valuation
- Since risk-neutral valuation is simplest, use it.

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Black/Scholes model

- Under risk-neutrality, current call value is:

$$c = e^{-rT} E(\tilde{c}_T)$$

Continuously compounded carry rate of call option.

Expected terminal call price based on truncated asset price distribution.

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Black/Scholes model

- Terminal call value is:

$$\tilde{c}_T = \begin{cases} \tilde{S}_T - X & \text{if } \tilde{S}_T - X > 0 \\ 0 & \text{if } \tilde{S}_T - X \leq 0 \end{cases}.$$

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Black/Scholes model

- Expected terminal value to right of X is:

$$E(\tilde{c}_T) = E(\tilde{S}_T \mid S_T > X)\,\Pr(S_T > X) - X\,\Pr(S_T > X)$$

  - Note: Does not depend on asset price distribution.

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Black/Scholes model

- Black-Scholes (1973) assume that asset price is log-normally distributed at option's expiration.

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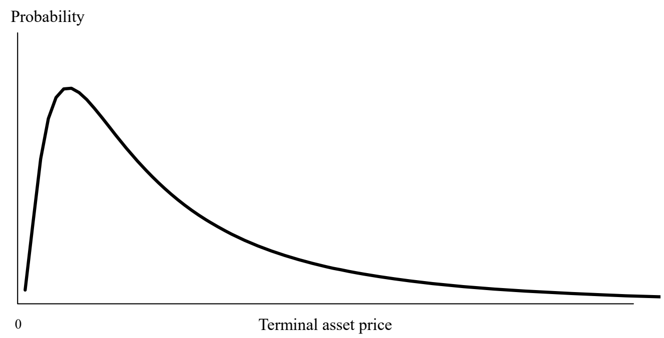
Black/Scholes model

- Log-normal distribution at option expiration.



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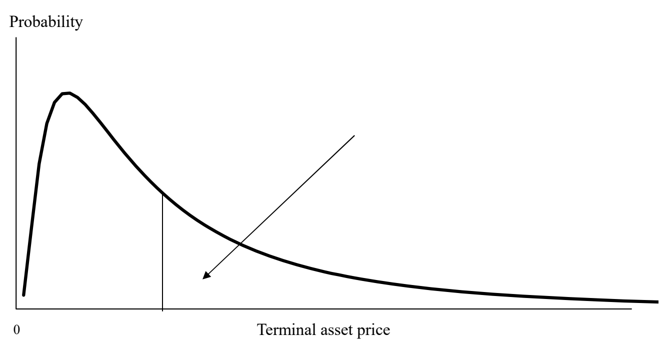
Black/Scholes model

- Expected terminal call value is based on area to right of exercise price *X*.



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Black/Scholes model

- Under log-normality, expected value is:

$$E(\tilde{c}_T) = Se^{bT}N(d_1) - XN(d_2)$$

where

$$d_1 = \frac{\ln\left(S/X\right)+\left(b+.5\sigma^2\right)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T},$$

$$b = r - i$$

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Black/Scholes model

- Under log-normality, expected value is:

$$E(\tilde{c}_T) = Se^{bT}N(d_1) - XN(d_2)$$

where $N(d_1)$ and $N(d_2)$: cumulative normal probabilities.

$$d_1 = \frac{\ln(S/X)+(b+.5\sigma^2)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

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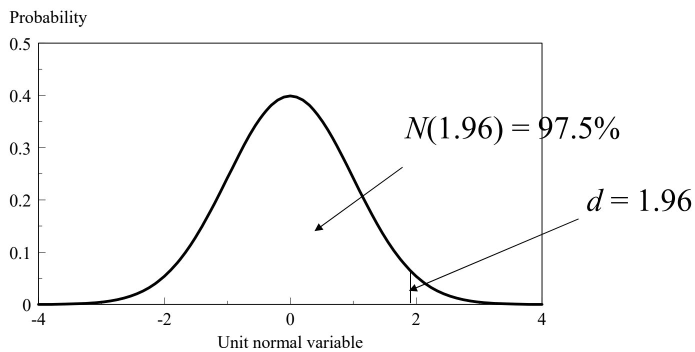
Black/Scholes model

- *N(d)* is cumulative normal probability.
  - Area under curve to left of *d*.



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Black/Scholes model

- Interpretation of terms.
  - Recall:

$$E(\tilde{c}_T) = E(\tilde{S}_T \mid S_T > X)\Pr(S_T > X) - X\Pr(S_T > X)$$

  - Compare with:

$$E(\tilde{c}_T) = Se^{bT}N(d_1) - XN(d_2)$$

  - Implies:

$$\Pr(S_T > X) = N(d_2).$$

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Black/Scholes model

- Interpretation of terms.
  - Recall:

$$E(\tilde{c}_T) = E(\tilde{S}_T \mid S_T > X)\Pr(S_T > X) - X\Pr(S_T > X)$$

  - Compare with:

$$E(\tilde{c}_T) = Se^{bT}N(d_1) - XN(d_2)$$

  - Implies:

$$E(\tilde{S}_T \mid S_T > X)\Pr(S_T > X) = Se^{bT}N(d_1)$$

Note: $N(d_1) \neq \Pr(S_T > X)$

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Black/Scholes model

- Take present value of expected future value to get European-style call formula,

$$c = Se^{(b-r)T}N(d_1) - Xe^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln(S/X) + (b + .5\sigma^2)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

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Black/Scholes model

- Formula is general and has many special cases.

$$c = Se^{(b-r)T} N(d_1) - Xe^{-rT} N(d_2)$$

where

$$d_1 = \frac{\ln(S / X) + (b + .5\sigma^2)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

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Stock option valuation

- Black and Scholes (1973) value call on non-dividend-paying stock.
  - Set $b=r$.

$$c = SN(d_1) - Xe^{-rT} N(d_2)$$

where

$$d_1 = \frac{\ln(S / X) + (r + .5\sigma^2)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

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Futures option valuation

- Black (1975) values call on futures.
  - Set b=0.

$$c = e^{-rT}[FN(d_1) - XN(d_2)]$$

where

$$d_1 = \frac{\ln(F/X) + .5\sigma^2 T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

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Stock index option valuation

- Merton (1973) values call on index.
  - Set $b = r - d$.

$$c = Se^{-dT}N(d_1) - Xe^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln(S/X) + (r - d + .5\sigma^2)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

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Currency option valuation

- Garman and Kohlhagen (1983) value call on foreign currency.
  - Set $b = r_d - r_f$.

$$c = Se^{-r_f T} N(d_1) - Xe^{-r_d T} N(d_2)$$

where

$$d_1 = \frac{\ln(S/X) + (r_d - r_f + .5\sigma^2)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

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Futures-style futures options

- Asay (1983) values futures-style call on futures.
  - Set $b = 0$ and $r = 0$.

$$c = FN(d_1) - XN(d_2)$$

where

$$d_1 = \frac{\ln(S/X) + .5\sigma^2 T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

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Black-Scholes/Merton (BSM) model

- BSM value of European-style call is:

$$c = Se^{(b-r)T} N(d_1) - Xe^{-rT} N(d_2)$$

where

$$d_1 = \frac{\ln(S/X) + (b + .5\sigma^2)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

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Black-Scholes/Merton (BSM) model

- Substitute:

$$b = r - i$$

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Black-Scholes/Merton (BSM) model

- BSM value of European-style call is:

$$c = Se^{-iT}N(d_1) - Xe^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln(S/X) + (r - i + .5\sigma^2)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

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Black-Scholes/Merton (BSM) model

- Standard call option is portfolio consisting of two simpler options.

$$c = Se^{-iT}N(d_1) - Xe^{-rT}N(d_2)$$
$$= \text{Asset-or-nothing call} - \text{Cash-or-nothing call}$$

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Black-Scholes/Merton (BSM) model

- Asset-or-nothing call has payoffs.

$$\tilde{c}_T^{AON} = \begin{cases} \tilde{S}_T & \text{if } \tilde{S}_T > X \\ 0 & \text{if } \tilde{S}_T \le X \end{cases}.$$

- Cash-or-nothing call has payoffs.

$$\tilde{c}_T^{CON} = \begin{cases} X & \text{if } \tilde{S}_T > X \\ 0 & \text{if } \tilde{S}_T \le X \end{cases}.$$

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Black-Scholes/Merton (BSM) model

- Buying standard call is like buying asset-or-nothing call and selling cash-or-nothing call.
- Under no-arbitrage principle,

$$\begin{aligned} c &= \text{Asset-or-nothing call} - \text{Cash-or-nothing call} \\ &= c^{AON} - c^{CON} \\ &= Se^{-iT}N(d_1) - Xe^{-rT}N(d_2) \end{aligned}$$

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Digression: Binary options

- Cash-or-nothing options are sometimes called binary options.
- Binary options traded on online exchanges.
  - Earliest in US was Iowa Electronic Market (IEM). Known as political outcome market.
    https://iemweb.biz.uiowa.edu/
  - PredictIt is operated by Victoria University of Wellington in New Zealand.
    https://www.predictit.org

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Stock option valuation

- Illustration:
  - Value call option with X=50 and T = 3 months.
  - Assume stock price is 52.50, stock pays no dividends, and stock's volatility rate is 35% annually.
  - Assume interest rate is 3% annually.

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Stock option valuation

- Intermediate computations:

$$d_1 = \frac{\ln\left[52.50/50e^{-.03(3/12)}\right]+.5(.35)^2(3/12)}{.35\sqrt{3/12}}$$
$$= .4092$$
$$d_2 = .4092 - .35\sqrt{3/12}$$
$$= .2342$$

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Stock option valuation

- Probability computations:

$$N(.4092) = .6588$$

$$N(.2342) = .5926$$

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Stock option valuation

- Option value computation:

$$c = SN(d_1) - Xe^{-rT}N(d_2)$$
$$= 52.50(.6588) - 50(.9925)(.5926)$$
$$= 5.179$$

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Stock option valuation

- OPTVAL function library: Compute option value.

OV_OPTION_VALUE(s,x,t,r,i,v,cp,ae)

s	underlying asset price
x	exercise price
t	time remaining until option's expiration
r	interest rate
i	income rate
v	volatility rate of underlying asset's returns
cp	call/put indicator "C" or "c" for call "P" or "p" for put
ae	American/European-style option indicator "A" or "a" for American-style "E" or "e" for European-style

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Stock option valuation

- Supporting file: Stock option valuation.xlsx.

Valuation of European-style options					
Stock		Parameter		Call	Put
Price (S)	52.5	$d_1/-d_1$		0.4092	-0.4092
Dividend yield (i)	0%	$d_2/-d_2$		0.2342	-0.2342
Volatility rate (σ)	35%				
		$N(d_1)$		0.6588	0.3412
Option		$N(d_2)$		0.5926	0.4074
Exercise price (X)	50				
Time to expiration (T)	0.25	Sexp(-iT)		52.500	52.500
		Xexp(-rT)		49.626	49.626
Market					
Interest rate (r)	3%	Value		5.179	2.306

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Black-Scholes/Merton (BSM) model

- Put formula follows directly from:
  - Put-call parity
  - Call formula

$$
\begin{aligned}
p &= Xe^{-rT} - Se^{-iT} + c \\
&= Xe^{-rT} - Se^{-iT} + Se^{-iT}N(d_1) - Xe^{-rT}N(d_2) \\
&= Xe^{-rT}\left[1 - N(d_2)\right] - Se^{-iT}\left[1 - N(d_1)\right] \\
&= Xe^{-rT}N(-d_2) - Se^{-iT}N(-d_1)
\end{aligned}
$$

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Black-Scholes/Merton (BSM) model

- European-style put formula is:

$$p = Xe^{-rT}N(-d_2) - Se^{-iT}N(-d_1)$$

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Black-Scholes/Merton (BSM) model

- Standard European-style put is a portfolio consisting of two simpler options.

$$p = Xe^{-rT}N(-d_2) - Se^{-iT}N(-d_1)$$
$$= \text{Cash-or-nothing put} - \text{Asset-or-nothing put}$$

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Black-Scholes/Merton (BSM) model

- Asset-or-nothing put has payoffs.

$$\tilde{p}_T^{AON} = \begin{cases} \tilde{S}_T & \text{if } \tilde{S}_T < X \\ 0 & \text{if } \tilde{S}_T \geq X \end{cases}.$$

- Cash-or-nothing put has payoffs.

$$\tilde{p}_T^{CON} = \begin{cases} X & \text{if } \tilde{S}_T < X \\ 0 & \text{if } \tilde{S}_T \geq X \end{cases}.$$

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Black-Scholes/Merton (BSM) model

- Buying standard put is like buying a cash-or-nothing put and selling an asset-or-nothing put.
- Under no-arbitrage principle,

$$\begin{aligned} p &= \text{Cash-or-nothing put} - \text{Asset-or-nothing put} \\ &= p^{CON} - p^{AON} \\ &= Xe^{-rT}N(-d_2) - Se^{-iT}N(-d_1) \end{aligned}$$

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Stock option valuation

- Illustration:
  - Value put option with X=50 and T = 3 months.
  - Assume stock price is 52.50, stock pays no dividends, and stock's volatility rate is 35% annually.
  - Assume interest rate is 3% annually.

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Stock option valuation

- OPTVAL function library: Compute put option value.

F12 =OV_OPTION_VALUE(B3,B8,B9,B12,B4,B5,"p","E")

Valuation of European-style options				
Stock		Parameter	Call	Put
Price (S)	52.5	d_1/-d_1	0.4092	-0.4092
Dividend yield (i)	0%	d_2/-d_2	0.2342	-0.2342
Volatility rate (σ)	35%			
		N(d_1)	0.6588	0.3412
Option		N(d_2)	0.5926	0.4074
Exercise price (X)	50			
Time to expiration (T)	0.25	Sexp(-iT)	52.500	52.500
		Xexp(-rT)	49.626	49.626
Market				
Interest rate (r)	3%	Value	5.179	2.306

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Stock option valuation

- OPTVAL function library: Compute put option value as difference between cash-or-nothing and asset-or-nothing puts.

OV_NS_ALL_OR_NOTHING_OPTION(s,x,t,r,i,v,cp,ac)

s	asset price
x	threshold price
t	time to expiration
r	interest rate
i	income rate
v	volatility rate
cp	call/put indicator "C" or "c" for call "P" or "p" for put
ac	asset-or-nothing/cash-or-nothing option indicator "A" or "a" for asset-or-nothing "C" or "c" for cash-or-nothing

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Stock option valuation

- OPTVAL function library: Compute put option value as difference between cash-or-nothing and asset-or-nothing puts.

F3 `=$B$3*OV_NS_ALL_OR_NOTHING_OPTION($B$3,$B$8,$B$9,$B$12,$B$4,$B$5,"P","A")`

	A	B	C	D	E	F
1	Valuation of or-nothing options					
2	Stock			Type	Call	Put
3	Price (S)	52.5		Asset-or-nothing	34.586	17.914
4	Dividend yield (i)	0%		Cash-or-nothing	29.407	20.219
5	Volatility rate (σ)	35%				
6				Value	5.179	2.306
7	Option					
8	Exercise price (X)	50				
9	Time to expiration (T)	0.25				
10						
11	Market					
12	Interest rate (r)	3%				

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Lesson summary

- Three equivalent approaches to option valuation:
  - Risk-free hedge portfolio
  - Risk-neutral valuation
  - Risk-averse valuation

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Lesson summary

- One key result is BSM call option formula.

$$c = Se^{-iT}N(d_1) - Xe^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln(S/X) + (r - i + .5\sigma^2)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

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Lesson summary

- Use risk-neutrality and log-normal asset price distribution to value European-style options.
  - Black/Scholes stock option formula
  - Black futures option formula
  - Merton constant dividend yield option formula
  - Garman/Kohlhagen currency option formula
  - Asay futures-style futures option model

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Lesson summary

- Standard options are portfolios of simpler options.
  - Asset-or-nothing calls and puts
  - Cash-or-nothing calls and puts

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Lesson summary

- All results depend upon assumption asset prices are log-normally distributed with constant volatility.
  - Implies log returns are normally distributed.
    - Stock returns have fat tails.
    - Stock index returns are skewed to left.
  - Is volatility stationary through time?

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