

Bond valuation mechanics

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Bond valuation mechanics

- Purpose:
 - Discuss default risk-free bond valuation (e.g., US Treasuries).
 - Inseparable from zero-coupon yield curve.
 - Distinguish between *spot* and *forward* interest rates.
 - Interest rates are continuously compounded.
 - I.e., prices of financial assets grow continuously through time, not at discrete intervals.
 - Define discount bond.
 - Develop measure interest rate risk called *duration*.
 - Compute forward rates based on zero-coupon yield curve.

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Definitions

r_i = spot interest rate on loan that begins today and is repaid at time t_i

$f_{i,j}$ = forward interest rate on loan that begins at time t_i and is repaid at time t_j

$DF_i = e^{-rt_i}$ = discount factor or price of \$1 repaid at time t_i

C_i = cash amount paid at time t_i

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Value of discount bond

- Value of discount bond is

$$B_{d,i} = C_i e^{-rt_i}$$

- Also called *strip* or *zero-coupon* bond.

- Implied spot rate of interest is

$$r_i = \frac{\ln(C_i / B_{d,i})}{t_i}$$

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Duration of discount bond

- Duration of discount bond is

$$DUR_{d,i} = \frac{\% \Delta \text{ bond price}}{\text{Change in yield}} = \frac{dB_{d,i} / B_{d,i}}{dr_i} = -\frac{t_i C_i e^{-r_i t_i}}{C_i e^{-r_i t_i}} = -t_i$$

- E.g., if duration of discount bond is 0.25 (3 months), one percent shift in yield will cause bond price to fall by 1/4%.

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Value of coupon-bearing bond

- Value of coupon-bearing bond is sum of values of portfolio of n discount bonds.

$$B_c = \sum_{i=1}^n B_{d,i} = \sum_{i=1}^n C_i e^{-r_i t_i}$$

- Importance of each discount bond in terms of overall value is

$$w_i = \frac{C_i e^{-r_i t_i}}{\sum_{i=1}^n C_i e^{-r_i t_i}} = \frac{B_{d,i}}{B_c} \quad \text{where} \quad \sum_{i=1}^n w_i = \frac{\sum_{i=1}^n B_{d,i}}{B_c} = 1$$

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Duration of coupon-bearing bond

- Duration of coupon-bearing bond is
 - Weighted-average duration of discount bonds
 - Weighted-average term to maturity of discount bonds

$$DUR_c = \frac{dB_c / B_c}{dr} = \sum_{i=1}^n w_i DUR_{d,i} = - \sum_{i=1}^n w_i t_i$$

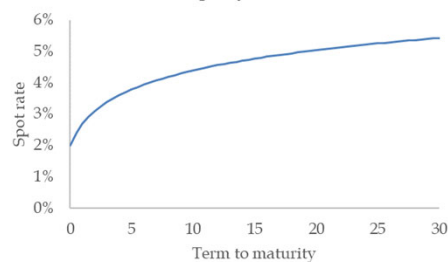
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Value/duration of CBB

- Illustration 1: Compute value and duration of 8% coupon bond with 30 years to maturity.
 - Assume zero-coupon yield curve is

$$r_t = 0.02 + 0.01 \ln(1+t)$$

Zero-coupon yield curve



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Value/duration of CBB

- Support file: Bond valuation mechanics.xlsx.
 - Sheet: ZC Valuing coupon bond

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Implied forward rate

- *Forward rates* are implied by observable zero-coupon yield curve.
 - Recall forward rate is rate on loan that begins at time t_i and matures at time t_j .
 - Observe r_i and r_j from zero-coupon term structure.
 - Compute implied forward rate $f_{i,j}$ using

$$e^{r_j t_j} = e^{r_i t_i} e^{f_{i,j}(t_j - t_i)}$$

- Assumptions?

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Computation of implied forward rate

- Start with

$$e^{r_j t_j} = e^{r_i t_i} e^{f_{i,j}(t_j - t_i)}$$

- Take log of both sides.

$$r_j t_j = r_i t_i + f_{i,j}(t_j - t_i)$$

- Rearrange.

$$f_{i,j} = \frac{r_j T_j - r_i T_i}{T_j - T_i}$$

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Forward loan

- Illustration 2:

- Compute implied 91-day borrowing rate on loan beginning in 59 days. Assume you
 - Buy December 28, 2023 T-bill at ask price.
 - Sell March 28, 2024 T-bill at bid price.
 - Support file: Bond valuation mechanics.xlsx
 - Sheet: Forward loan mechanics

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Forward loan

□ Illustration 2:

- Cash flows are:

Action	T-bill maturity	Price	No. of units	Dates		
				20231228	20231228	20240328
Buy	20231228	99.1314	1	-99.1314	100.0000	
Sell	20240328	97.7833	-1.01379	99.1314		-101.3786
Net				0.0000	100.0000	-101.3786

No investment Receive Pay

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Forward loan

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Net				0.0000	100.0000	-101.3786

$$\text{Borrowing rate from 12/28/23 to 3/28/24} = \frac{\ln(101.3786 / 100)}{91 / 365} = 5.492\%$$

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Lesson summary

- Defined and explained:
 - Spot rates and how implied spot rates are computed.
 - Zero-coupon yield curve and how to estimate it.
 - Forward rates and how implied spot rates are computed.
 - Discount factors and how discount factors are computed.
- Showed how to value and compute duration of:
 - Discount bonds
 - Coupon-bearing bonds

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