*Roman L. Weil**

Macaulay's Duration: An Appreciation

Nineteen seventy-three is the thirty-fifth anniversary of the publication of Frederick R. Macaulay's monumental study[1] of railroad bond prices. In that study, Macaulay proposed the measure *duration* to represent the "average" maturity of a stream of payments—such as, for example, a bond. His measure, or measures not substantially different from it, has often been used in economics, finance, and insurance. The increasing use of duration and of duration-related measures led me, on the occasion of the thirty-fifth anniversary, to put together these notes on the history of duration in the hope that future scholars, otherwise unaware of duration or of its uses in the literature, will be saved time and error.

Macaulay defined the duration of a stream of payments as follows. Let S_{t_j} represent the future value of a payment to be received t_j units of time hence, and let P_{t_j} represent its present value. Then the duration of a stream of payments $(S_{t_1}, S_{t_2}, \ldots, S_{t_n})$ with present values $(P_{t_1}, P_{t_2}, \ldots, P_{t_n})$ is

$$D = \sum_{i=1}^{n} t_i P_{t_i} \Big/ \sum_{i=1}^{n} P_{t_i}.$$

The measure has dimension time and is, in a sense, equal to the period of time which elapses before the "average" dollar of present value from a stream of payments is received. The duration of a stream may be thought of as the average life of the stream. Duration has interesting properties.[2] Note, for example, that the duration of a stream of positive payments is always less than the time until the last payment, unless the "stream" is a single payment. Note, too, that the duration of an ordinary coupon bond is an increasing function of the bond's maturity if and only if the bond sells at or above par.[3] The duration of a perpetuity in

* Associate professor of management and information sciences, Graduate School of Business and Committee on Information Sciences, University of Chicago. Lawrence Fisher contributed to my understanding of these problems in a way only those who know him can fully comprehend. Thanks. The National Science Foundation supported this work.

1. Frederick R. Macaulay, *Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields, and Stock Prices in the United States since 1856* (New York: Columbia University Press for the National Bureau of Economic Research, 1938).

2. See Macaulay, pp. 45 ff., for a full development.

3. The durations for coupon bonds with various coupon rates and various yields to maturity are given in table 4 of Lawrence Fisher and Roman L. Weil, "Coping with the Risk of Interest-Rate Fluctuations: Returns to Bondholders from Naïve and Optimal Strategies," *Journal of Business* 44, no. 4 (October 1971): 408–31.

arrears is relatively short—equal to $(p + r)/rp = (1 + r/p)/r$ years, where r is the discount rate per period and p is the number of (equally spaced) payments per period.

Macaulay wanted a scalar measure that sensibly indicated the (time) length of a bond. By example and counterexample, he proposed and discarded various measures until he described duration. He showed that the measure behaved the way he wanted it to and derived properties such as those mentioned above.

Hicks published *Value and Capital*[4] in 1939, 1 year after Macaulay's book appeared. Hicks defined and used "an elasticity [of a capital value] with respect to a discount ratio [i.e., factor]" that is equivalent to Macaulay's duration.[5] Hicks called his measure "average period." Hicks used his measure[6] to make concrete the intuitive notion that, when interest rates fall, producers will substitute money (or the capital it can buy) for other means of production and that average period of production plans increases.

Macaulay wanted a measure of time; Hicks, an elasticity. They derived the same measure. Hicks noted that, although elasticities are ordinarily "pure" numbers, this particular one had dimension time, and he explained why. Grove[7] appears to have been the first to cite both Macaulay and Hicks. Fisher,[8] then unaware of Hicks's interest elasticity, showed that Macaulay's measure had the properties of an elasticity.

Not all writers who use duration or duration-like measures cite any authority. As far as I can ascertain, few cite Hicks. No doubt Hicks discovered the measure independently of Macaulay.

In 1945, Samuelson,[9] then unaware of Macaulay's work,[10] analyzed the effect of interest rate changes on institutions such as universities, insurance companies, and banks. He developed a measure—the "weighted average time period of payments"—essentially equivalent to duration, and proved, in effect, that, if the duration of an institution's assets is larger (smaller) than that of its liabilities, then the institution will lose (profit) when interest rates rise and profit (lose) when interest rates fall.[11]

In 1952, Redington, in a paper not well known by economists, de-

4. J. R. Hicks, *Value and Capital* (Oxford: Clarendon Press, 1939).
5. See Hicks, p. 186.
6. See Hicks, chap. 17.
7. Myron A. Grove, "A Model of the Maturity Profile of the Balance Sheet," *Metroeconomica* 18, no. 1 (April 1966): 40–55.
8. Lawrence Fisher, "An Algorithm for Finding Exact Rates of Return," *Journal of Business* 39, no. 1, pt. 2 (January 1966): 111–18. A draft of this article was available at the University of Chicago as early as 1963.
9. Paul A. Samuelson, "The Effects of Interest Rate Increases on the Banking System," *American Economic Review* 35 (March 1945): 16–27.
10. Samuelson wrote this in a letter to me.
11. Samuelson concluded that the impending postwar increase in interest rates would benefit banks whose liabilities are generally of shorter duration than its assets. His formula in footnote 1 contains an error: the term "$\log_e(1 + i)$" should not appear.

fined the mean term of an asset stream and of a liability stream.[12] He proved that the profits of an insurance company were immune (could not be reduced but might be increased) to any small (infinitesimal) change in interest rates provided that the mean term of assets equaled the mean term of liabilities. Wallas[13] provides an expanded exposition of Redington's brilliant contribution.

Redington writes as though, at that time, the best investment practice for insurance companies was to match asset and liability streams, period by period. (Such a matching would, of course, achieve equality of asset and liability mean terms or durations.) Interestingly enough, the first paper I have seen on the matching of insurance company assets and liabilities to achieve immunization was written in 1942 by Tjalling C. Koopmans[14] when he worked for Penn Mutual Life Insurance Company. Although this paper has not been published, a surprising (to me) number of economists know about it.

Durand, in 1957, argued that the only financial assets with super-long durations were growth stocks, so that institutions with long-duration liabilities will want to hold growth stocks to reduce risk of loss from fluctuations in interest rates.[15]

Fisher showed in 1966 that $dV/di = -D/V$, where V is the present value of a stream of payments, i the interest rate used for continuous discounting, and D the duration of the stream. He then used this result to derive a gradient step in a Newton-Raphson method for efficiently calculating exact rates of return on streams of arbitrarily spaced payments. Hicks's formulation used $dV/d(1+r)$, where r is the rate compounded periodically. Fisher's interest elasticity is proportional to Hicks's accumulation-factor elasticity.

Whittaker found duration useful in his analysis of British unit trusts (mutual funds) and wrote a note on the measure itself.[16]

Fisher and Weil used duration to develop an optimal strategy for

12. F. M. Redington, "Review of the Principles of Life-Office Valuations," *Journal of the Institute of Actuaries* 78, no. 3 (1952): 286–315; followed by his "Abstract of Discussion," pp. 316–40. Redington's definition of mean term of a sequence S with present values $(P_{t_1}, \ldots, P_{t_n})$ is

$$\sum_{i=1}^{n} t_i P_{t_i}.$$

13. G. E. Wallas, "Immunization," *Journal of the Institute of Actuaries Students' Societies* 15 (1960): 345–57.

14. Tjalling C. Koopmans, *The Risk of Interest Fluctuations in Life Insurance Companies* (Philadelphia: Penn Mutual Life Insurance Co. 1942).

15. David Durand, "Growth Stocks and the Petersburg Paradox," *Journal of Finance* 12, no. 3 (September 1957): 348–63. Durand attributed to Wilfred Perks remarks that I believe should have been attributed to A. F. Murray. See the laudatory discussion following Redington's article.

16. John Whittaker, "Minimizing the Burden of the Dollar Premium," *Investment Analyst* (October 1969), pp. 26–33; and "The Relevance of Duration," *Journal of Business Finance* 2 (Spring 1970): 1–8.

bond investment to achieve a near-riskless asset and to measure returns to bondholders. Further, we extended Redington's result on immunization to *large* interest rate fluctuations of certain kinds.

Perhaps others have used duration or similar concepts. Several authors could have benefited from knowing about it. Wehrle[17] concluded, incorrectly, I think, that insurance companies would most prefer a government bond with "a 50–100 year maturity, a 3.5 to 3.9 percent coupon and be noncallable and preferably a 'tap' issue. The 'tap' issue means the security would be available for purchase from the Treasury for an extended period." The duration of a semiannual 3.5 percent coupon bond with 100 years to maturity is only slightly larger than 17 years when interest rates are such that the bond yield is 6 percent. Such an instrument would not much help a firm that wanted to achieve immunization and had liabilities with a long duration. A long-horizon, single-payment note would provide a large-duration instrument.[18] Hopewell and Kaufman,[19] in a review of duration and its implications, show how Malkiel[20] would have benefited from knowing of duration and its properties.

Duration has been a powerful concept. I hope this note will help those who might be otherwise unaware of what can be done with it.[21]

17. Leroy S. Wehrle, "Life Insurance Investment—the Experience of Four Companies," *Yale Economic Essays* 1 (1961): 70–136.

18. See Fisher and Weil (n. 3 above), p. 419.

19. M. H. Hopewell and G. C. Kaufman, "Bond Price Volatility and Term to Maturity: A Generalized Respecification," *American Economic Review*, forthcoming.

20. Burton H. Malkiel, *The Term Structure of Interest Rates* (Princeton, N.J.: Princeton University Press, 1966).

21. William A. Brock of the University of Chicago has proved that, up to a constant, Macaulay's definition is the only one that satisfies a set of axioms or properties one wants the measure to have. M. A. Grove of the University of Oregon writes me that he has done a paper "on 'immunization' in the Arrow framework" and that "Hickman in his unpublished manuscript 'The Term Structure of Interest Rates' also makes use of duration at several points. . . ." David Durand of M.I.T. has prepared two articles that use duration to help resolve various questions concerning the time dimension of investments: "Payout Period, Time Spread, and Duration: Aids to Judgment in Capital Budgeting," submitted to the *Journal of Bank Research*, and "Time as a Dimension of Investment."