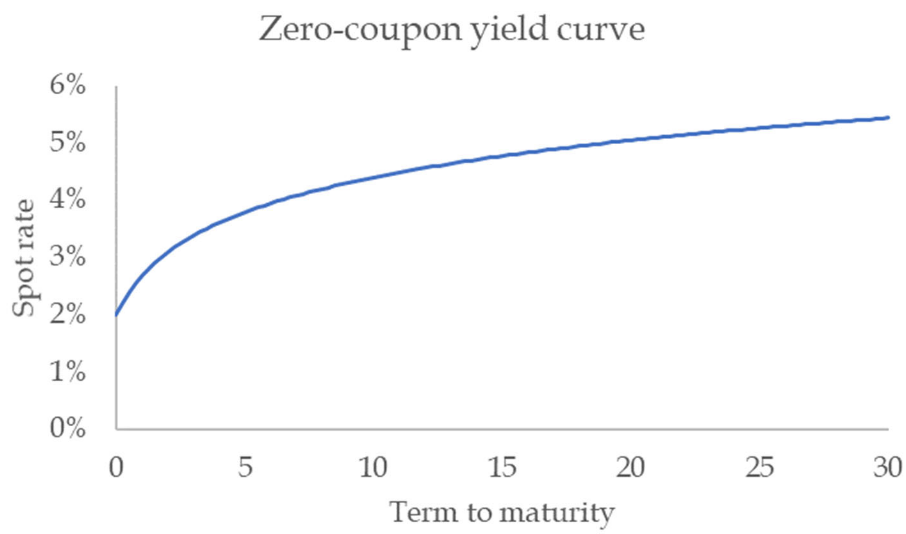
BOND VALUATION MECHANICS

Spot rates

A *spot rate of interest* is the interest rate on a loan that starts today and ends with a single payment at a pre-specified time in the future. The *term structure of (spot) interest rates* refer to the relation between the promised yield of a *discount bond* (i.e., a zero-coupon bond) and term to maturity for bonds with the same degree of default risk (e.g., U.S. Treasuries). Often-called the *zero-coupon yield curve,* it is usually upward-sloping as illustrated below. If the one-year (continuously compounded) spot rate is 2.69%, borrowing 1 today means repaying $e^{0.0269(1)} = 1.02730$ in one year. If the two-year rate is 3.10%, borrowing 1 today means repaying $e^{0.0310(2)} = 1.0639$ in two years, and so on.



Discount bonds

A *discount bond* pays a single cash flow at time t_i. Its valuation formula is:

$$B_{d,i} = C_i e^{-r_i t_i}$$

where C_i is the cash amount (i.e., face value) of discount bond *i* to be received at time t_i, and r_i is the promised yield to maturity.

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Coupon-bearing bond

A *coupon-bearing bond* makes *n* periodic (e.g., every six months) cash payments (i.e., coupons), where the n^{th} payment includes the bond's repayment of principal (i.e., face value). If viewed as a collection of payments, a coupon-bearing bond is nothing more than a portfolio of discount bonds maturing at different points in time. Thus, the coupon-bearing bond valuation formula is:

$$B_c = \sum_{i=1}^{n} B_{d,i} = \sum_{i=1}^{n} C_i e^{-r_i t_i}$$

where C_i is the cash payment at time t_i and n is the number of cash payments. The importance of each payment (i.e., its weight) in determining the overall value of the bond is:

$$w_i = \frac{C_i e^{-r_i t_i}}{\sum_{i=1}^{n} C_i e^{-r_i t_i}} = \frac{B_{d,i}}{B_c},$$

where

$$\sum_{i=1}^{n} w_i = \frac{\sum_{i=1}^{n} B_{d,i}}{B_c} = 1.$$

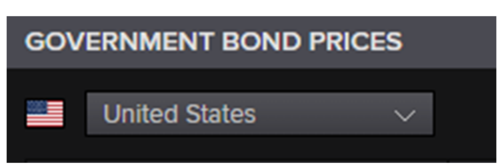
Estimation of zero-coupon yield curve

As a practical matter, opportunities to estimate the zero-coupon yield curve are rare because seldom is it the case that an array of bonds, let alone zero-coupon bonds, have the same level of default risk. The exception is, of course, U.S. Treasuries bills, notes, and bonds, which have maturities extending out as long as 30 years. The Excel file, Bond valuation mechanics.xlsx, contains the zero-coupon yield curve based on the prices of Treasury bills and strip bonds at the close of trading on October 30, 2023.

Before discussing the results, I will describe the procedure that I used for downloading the data.[1] The source is Refinitiv Workspace.

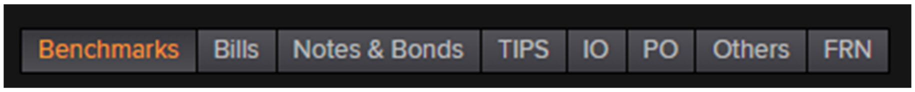
Steps:

1) Type GOV (for Government bonds) in the search bar.
2) Select United States as the government in the upper left.



[1] I thank AJ Reams and HD McKay for developing the procedure to access the data.

3) About halfway down the page, you will see an assortment of tabs.



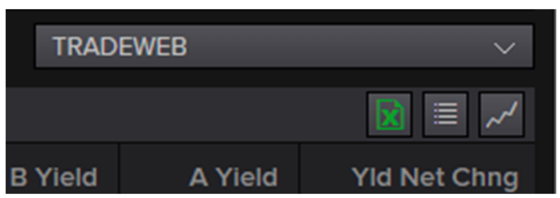
These are the different Treasury securities available. The first five are:

a) Bills – Treasury bills (discount bonds) with maturities up to 1 year
b) Notes & Bonds – Coupon-bearing Treasury notes and bonds with maturities up to 30 years.
c) TIPS – Treasury inflation-protected securities
d) IO – interest-only (i.e., coupon-only) strip bonds
e) PO – principal-only strip bonds

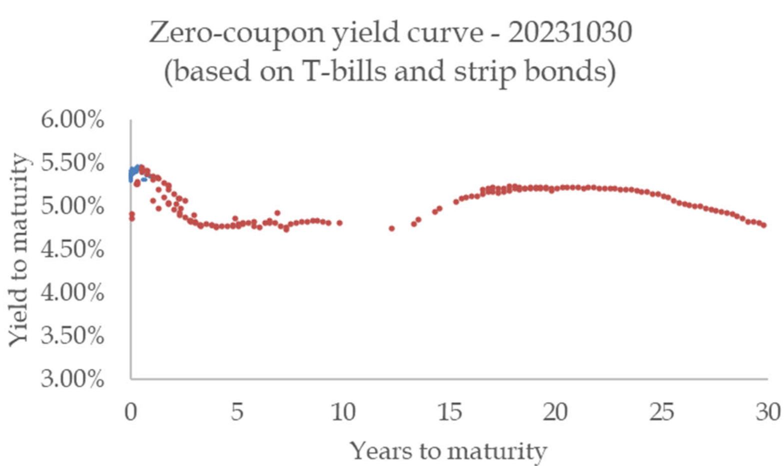
Note: d) and e) are not issued as original-issue discount bonds. Different financial institutions, brokers and dealers engage in the practice of making markets in each of the payments of a coupon-bearing Treasury bond as IO or PO securities. In the absence of costless arbitrage opportunities, the sum of the IO and PO prices of payments of a coupon-bearing bond should be equal to the price of the coupon-bearing bond.

https://www.treasurydirect.gov/marketable-securities/understanding-pricing/

4) Download the prices of the Bills and the POs by clicking the Excel button in the upper right corner of the table.



The Excel file shows that the zero-coupon yield curve based on the T-bills and strip bonds is:



From an historical perspective, the shape of this yield is unusual in two respects.

1) The curve slopes downward for maturities up to 10 years.
2) For maturities greater than 10 years, it appears humped.

Typically, it is monotonically upward sloping. Lenders like the liquidity of short-term loans but are willing to lend long if provided with a high enough liquidity risk premium.

Promised yield to maturity

Promised yield to maturity is a convenient, descriptive statistic meant to reflect the promised rate of return on the bond if held to maturity. If the observed price of the bond is set equal to the present value of the promised cash flows, the yield to maturity can be determined (iteratively) using the equation,

$$B_c = \sum_{i=1}^{n} C_i e^{-yt_i} .$$

Duration

Duration is a concept developed by Macaulay in his study of railroad bonds in 1930. Macauley was trying to capture was the sensitivity of bond price to a shift in the yield curve. To develop such as measure, take the derivative of the bond valuation formula with respect to yield,

$$\frac{dB_c}{dy} = -\sum_{i=1}^{n} t_t C_i e^{-yt_i}$$

and then divide by the bond value,

$$\frac{dB_c / B_c}{dy} = -\sum_{i=1}^{n} t_i \left(\frac{C_i e^{-yt_i}}{B_c} \right) = -\sum_{i=1}^{n} t_i w_i \equiv -D_c .$$

As the formula shows, D_c is a descriptive statistic that measures the percent change in bond price for a given change in yield to maturity. It is also the weighted average of the term to maturity of the bond's constituent discount bonds since the weights sum to one,

$$\sum_{i=1}^{n} w_i = 1 .$$

A variety of other definitions of duration exist. A criticism of the above version is that it assumes an additive shift in the entire yield curve. It is more common to see zero-coupon rates shift more dramatically at the short end than the long end. The key is being able to model the shift (e.g., a multiplicative shift). If the character of the shift is known, the *curve duration* can be determined as

$$\frac{dB_c / B_c}{dr_i} = -\sum_{i=1}^{n} t_i \left(\frac{C_i e^{-r_i t_i}}{B_c} \right) = -\sum_{i=1}^{n} t_i w_i \equiv -D_c^r .$$

Note that the duration of a discount bond is simply minus its stated time to maturity, that is,

$$\frac{dB_{d,i} / B_{d,i}}{dr_i} = -t_i \left(\frac{C_i e^{-r_i t_i}}{C_i e^{-r_i t_i}} \right) = -t_i .$$

Hence, as noted above, the duration of a coupon-bearing bond is a weighted average time to maturity of the bond's cash payments, where weights equal the present values of the payments divided by the bond's price.

Oftentimes bonds contain embedded options (e.g., callable bonds or extendible bonds) that effectively shorten or lengthen the bond's cash payment stream. In these instances, the term, ef*fective duration*, applies. A universal formula for effective duration is not possible because the nature of the embedded option is bond specific. Estimating this measure empirically is possible by regressing the daily returns of a bond *i* on a known bond index, *M*,

$$R_{i,t} = \alpha_i + \beta_i R_{M,t} + \varepsilon_{i,t}$$

over a short sample period. The intuition is that the shift in the yield curve each day induces movements in the bond price and the bond index level. The estimate of β_i is the effective duration of the relative to the effective duration of the index.[2]

Forward rates

Closely related to the concept of the zero-coupon yield curve is the concept of *implied forward rates.* A *forward rate* is the interest rate on a loan that begins in the future (e.g., a one-year loan that will begin in one year). To understand implied forward rates, consider the following illustration. Suppose you want to invest \$1 for two years with zero default risk. The zero-coupon yield curve says you can do so at a rate of 3.10%, and you will have \$1.0639 at the end of two years. But you can also deposit \$1 in a one-year loan at a rate of 2.69%, and then reinvest the proceeds in a one-year loan in one year. If individuals are risk-neutral, the relation,

$$e^{0.0310\times 2} = e^{0.0269\times 1} e^{f_{1,1}\times 1} ,$$

must hold in the absence of costless arbitrage opportunities. Solving for the implied one-year forward rate in one year, $f_{1,1} = 3.50\%$. In other words, the current term structure of spot rates (i.e., the zero-coupon yield curve) implies that the one-year forward rate on a one-year loan is 3.50%. The generalized formula for implied forward rates is

[2] Technically, the intercept term in this regression should be zero.

$$f_{i,j} = \frac{r_j t_j - r_i t_i}{t_j - t_i}$$

where $f_{i,j}$ is the implied forward rate of interest on a loan beginning at time t_i, and ending at time t_j.

References

Macaulay, Frederick R., 1938, *Some Theoretical Problems Suggested by Movements of Interest Rates, Bond Yields and Stock Prices in the United States since 1856,* (New York: Columbia University Press for the National Bureau of Economic Research).

Weil, Roman L., 1973, Macaulay's duration: An appreciation, *Journal of Business* 46, 589-592.